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Dynamic Programming and Inventory Control

Alain Bensoussan

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Dynamic Programming and Inventory Control

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CHAPTER 1

INTRODUCTION

The objective of this book is to present a unified theory of Dynamic Programming and Markov Decision Processes with its application to a major field of Operations Research and Operations Management, Inventory Control. We will develop models in discrete time as well as in continuous time. In continuous time, the diversity of situations is huge. We will not cover all of them and concentrate on the models of interest to Inventory Control. In discrete time we will focus mostly on infinite horizon models. This is also the situation when the Bellman equation of Dynamic Programming is really a functional equation, the solution of which is called the value function. With a finite horizon, Dynamic Programming leads to a recursive relation, which is an easier situation. Of course, finite horizon problems can also be considered as approximations to infinite horizon problems, and this will be used in our presentation. When the horizon is infinite, the discount plays an essential role. An important problem concerns the behavior of the value function when the discount factor tends to 1. The value function also tends to infinity, but an average cost function becomes meaningful. This development is called ergodic theory, an interesting aspect of which is that the solution, when it exists, can be simpler than in the case when the discount is smaller than 1. However, the theory is more complex. The simplicity is linked to the fact that the problem becomes static, instead of dynamic. The complexity stems from the fact that this static problem is an averaging. The averaging procedure is not trivial and may be not intuitive.

Another important question concerns the difference between impulse control and continuous control. The difference is particularly understandable in continuous time. An impulse control will elicit an instantaneous jump for the state, whereas a continuous control can only lead to a continuous evolution of the state. In practice, this occurrence is linked to fixed costs, namely costs which arise just because a decision is taken, whatever the impact of the decision may be. In discrete time, the difference disappears in principle, since time is not continuous by construction. However, fixed costs remain. The consequence is that an appropriate formulation of impulse control remains meaningful and useful in discrete time. Indeed, in discrete time, the usual assumption is that the result of a decision materializes at the end of the period, whereas the decision is taken at the beginning of the period. Impulse control in discrete time means that the result also materializes at the beginning of the period, so instantaneously. We will also consider ergodic control in the context of impulse control and justify some simple rules which are currently used in practice.

In chapter 2, we shall introduce some of the classical static problems, which are preliminary to the dynamic models of interest in inventory control. By static, we mean that we have to solve an ordinary optimization problem. The decision does not depend on time. Such models occur when one considers one period only, or when one assumes that the periods reproduce in a periodic way. We shall revisit the periodic models later to check their optimality in stationary situations.

Although there is a huge literature on these domains, we believe that this book bears major differences. It is not a general text on Control Theory and Dynamic Programming, in the sense that the system's dynamics is mostly limited to inventory models. On the other hand, for these models it tries to be as comprehensive as possible. We do not however develop finite horizon models in discrete time, since they are largely described in the existing literature. On the other hand, the ergodic control problem is considered in great detail, and probabilistic proofs as well as analytical proofs are provided. As already mentioned, this model is extremely important in practice, since it is analogous to a static optimization problem. However, the literature is limited as far rigorous treatment is concerned. The techniques developed in this work can be extended to more complex models, covering additional aspects of inventory control. So many interesting research topics can be formulated from the contents of this book.

This book has benefited from my teaching at the University of Texas at Dallas, and exchanges with colleagues at both the University of Texas at Dallas and the Hong Kong Polytechnic University, and it has also greatly benefited from the support of the World Class University Program at Ajou University. Thanks to this support, it has been possible to focus on research and writing this book. Finally, I would like to thank Laser Yuan warmly for improving my LyX typesetting and IOS Press for publishing the book.

CHAPTER 2

STATIC PROBLEMS

2.1. NEWSVENDOR PROBLEM

This is the oldest problem in the domain. It illustrates a one period problem.

A newsvendor cannot carry over newspapers of the day to the next day. He orders q and faces a random demand D. This is a random variable with c.d.f (cumulative distribution function) $F(x)$. We note $\bar{F}(x) = 1 - F(x)$, $f(x) = \frac{dF}{dx}(x)$. He faces a left over cost when the order q is larger than the demand and a shortage cost when the order is less than the demand. So his objective function is

$$J(q) = hE(q - D)^+ + pE(D - q)^+, \tag{2.1.1}$$

where E represents the mathematical expectation.

The optimal q is clearly solution of

$$F(\hat{q}) = \frac{p}{h + p}. \tag{2.1.2}$$

Suppose in addition, that the newsvendor buys the newspaper at a unit cost c then the function $J(q)$ has to be changed into

$$J(q) = cq + hE(q - D)^+ + pE(D - q)^+. \tag{2.1.3}$$

We need to assume that $p > c$ otherwise there is no incentive to buy (the function $J(q)$ is then increasing). The optimal order is solution of

$$F(\hat{q}) = \frac{p - c}{h + p}. \tag{2.1.4}$$

If the cost is $K + cq$ representing the sum of a fixed cost (independent of the quantity) and a variable cost (proportional to the quantity) there is no change in the optimal decision, provided that one decides to order. In this case the amount K is a charge which is due anyway. However, since we have the possibility not to order, in which case the cost is hED, the Newsvendor must compare hED with $K + J(\hat{q})$. If the second number is higher than the first one, then he chooses not to order anything.

In formula (2.1.1) we consider implicitly that we compare the demand D to the order q at the end of the day (more generally the period). We may have a more accurate treatment assuming that the total demand D materializes with a uniform rate along the period. If the period length is 1 then the average left over is $\int_0^1 (q - Dt)^+ dt$ and the shortage is $\int_0^1 (q - Dt)^- dt$, so the objective becomes

$$J(q) = hE\int_0^1 (q - Dt)^+ dt + pE\int_0^1 (Dt - q)^+ dt, \tag{2.1.5}$$

hence

$$J(q) = h \int_0^1 tF\left(\frac{q}{t}\right) dt + p \int_0^1 t\bar{F}\left(\frac{q}{t}\right) dt,$$

and the optimal quantity is given by

$$F(\hat{q}) + \hat{q} \int_{\hat{q}}^{\infty} \frac{f(u)}{u} du = \frac{p}{h+p}. \tag{2.1.6}$$

It is easy to check that equation (2.1.6) has a unique solution.

2.2. EOQ MODEL

2.2.1. BASIC EOQ MODEL. This is also a famous and very old model. EOQ stands for Economic Order Quantity. It is due to F.W. Harris, [**23**] and R.H. Wilson, [**40**]. The model is deterministic. In fact the model is not static but dynamic and time is continuous. We will revisit it when we will treat dynamic models. However, it is possible to reduce it to a static model, by a simple argument of periodicity. The demand materializes uniformly (and permanently) at a rate per unit of time λ. We forbid the possibility of shortage. We order a quantity q when the inventory vanishes. Suppose we start at time 0 with q then at time T such that

$$q = \lambda T,$$

the inventory vanishes. Clearly, T can be taken as a cycle. We want to minimize the average cost during a cycle.

The level of inventory during the cycle is given by

$$y(t) = q - t\lambda.$$

The cost of buying q is given by $K + cq$, where K represents the fixed cost and c the variable cost. There is a storage cost proportional to the quantity held per unit of time. Therefore, during the cycle, the storage cost is given by

$$\int_0^T hy(t)dt = h \int_0^T (q - t\lambda)dt = \frac{hqT}{2}.$$

The total cost, during a cycle can then be written as

$$K + cq + \frac{hqT}{2} = T(c\lambda + C(q)),$$

with

$$C(q) = K\frac{\lambda}{q} + \frac{hq}{2}.$$

Therefore the optimal order is (EOQ Formula)

$$\hat{q} = \sqrt{\frac{2K\lambda}{h}}.$$

Note that

$$K\frac{\lambda}{\hat{q}} = \frac{h\hat{q}}{2},$$

and

$$C(\hat{q}) = h\hat{q}.$$

Recalling that the average cost per cycle contains also the constant λc, the optimal average cost is thus

$$h\hat{q} + \lambda c.$$

Note also that $\frac{q}{2}$ is the average inventory and the demand per unit of time is λ; this implies

$$\frac{\lambda}{\frac{q}{2}} = \text{rotation rate of the stock},$$

which is an important indicator of management. We see that the optimal rotation rate is $\sqrt{\frac{2\lambda h}{K}}$. Increasing this rate of rotation is possible if one decreases the cost K.

The next question is: Are we sure we act optimally following the EOQ formula? Are we sure it is optimal to take periodic decisions given by the EOQ formula. Although this looks natural and has been used for long, it is worth proving it. Moreover, the periodic decision does not take into account the possibility of interest rates. What are the changes if we discount the flow of costs? To answer correctly to this question requires the apparatus of impulse control, which will be described later.

2.2.2. TRANSFORMED EOQ FORMULA. Suppose now that there is no immediate delivery. Instead, the delivery is provided at a continuous rate r. Since there is no possibility of shortage, we must have $r \geq \lambda$. The evolution of the stock is given by the following formula

$$\begin{aligned} y(t) &= (r-\lambda)t, \forall t \leq T_0 = \frac{q}{r}, \\ y(t) &= q - \lambda t,\, T_0 < t \leq T = \frac{\lambda}{r}. \end{aligned}$$

One can then check that

$$\hat{q} = \sqrt{\frac{2K\lambda r}{h(r-\lambda)}}.$$

This formula leads to ∞ when $r = \lambda$. One must interpret it in the sense that there will be a continuous delivery at the level of the demand. The stock remains 0, and the cost per unit of time equal to $c\lambda$. There is no optimization in this case.

2.3. PRICE CONSIDERATIONS

We consider here that the cost of buying the product is composed of a fixed cost K, a variable cost cq, which are internal costs for the company. Besides there is a buying price per unit of product, which may depend on the amount purchased, when for instance, discounts are possible. We will denote this price by α. We also assume that the storage cost h is linked to α by the relation $h = i\alpha$.

So on a cycle, when the price is fixed, we get for an order q the following cost

$$K + q(c+\alpha) + i\alpha\frac{qT}{2}.$$

Per unit of time we obtain $C(q) + \lambda c$, with

$$C(q) = \frac{K\lambda}{q} + \left(i\frac{q}{2} + \lambda\right)\alpha.$$

In this framework, α is simply an additional cost to the variable cost c. We study now the effect of discounts.

2.3.1. EOQ FORMULA WITH UNIFORM PRICE DISCOUNT. Suppose we introduce the possibility of discount on the quantity. We assume that the price of the product is α_j, if the quantity ordered lies in the interval $[q_{j-1}, q_j)$. So we have the formula

$$C(q) = \frac{K\lambda}{q} + \left(i\frac{q}{2} + \lambda\right) \sum_j \alpha_j \mathbb{1}_{\{q_{j-1} \le q < q_j\}}.$$

We notice that the cost function is not continuous. Nevertheless we have the property:

Exercise 2.1. The minimum of $C(q)$ can be obtained. Identify it.

2.3.2. EOQ FORMULA WITH PROGRESSIVE DISCOUNT. Let us consider an extension which will lead to a continuous function, see V. Giard [**20**].

Suppose that, if $q_{j-1} \le q < q_j$, only the part $q - q_{j-1}$ is paid at price α_j. Let A_j be the cost of procuring the quantity q_j, then clearly

$$A_j = A_{j-1} + (q_j - q_{j-1})\alpha_j,\ A_0 = 0,$$

and for q as before, the full procurement cost is $A_{j-1}+(q-q_{j-1})\alpha_j$, and the average unit price is $\frac{A_{j-1}+(q-q_{j-1})\alpha_j}{q} = \alpha_j + \frac{B_j}{q}$, where

$$B_j = A_{j-1} - q_{j-1}\alpha_j,$$

hence

$$B_{j+1} = \sum_{h=1}^{j} q_h(\alpha_h - \alpha_{h+1}),\ B_1 = 0,$$

therefore, we have

$$C(q) = \frac{K\lambda}{q} + \left(i\frac{q}{2} + \lambda\right) \sum_j \left(\alpha_j + \frac{B_j}{1}\right) \mathbb{1}_{\{q_{j-1} \le q < q_j\}}.$$

This function is continuous, since

$$\alpha_j + \frac{B_j}{q_j} = \alpha_{j+1} + \frac{B_{j+1}}{q_j}.$$

If j runs from 1 to J, then we use the convention $q_{J+1} = +\infty$. Moreover α_{J+1} is the price of acquiring a quantity larger than q_J. So

$$C(q) = \frac{K\lambda}{q} + \left(i\frac{q}{2} + \lambda\right) \sum_{j=1}^{J+1} \left(\alpha_j + \frac{B_j}{q}\right) \mathbb{1}_{\{q_{j-1} \le q < q_j\}}.$$

Exercise 2.2. Check that

$$C(q) = \frac{K\lambda}{q} + \frac{i\alpha_1 q}{2} + \left(\frac{i}{2} + \frac{\lambda}{q}\right) \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j)(q - q_j)^+,$$

therefore $C(q)$ is piecewise differentiable and

$$C'(q) = -\frac{K\lambda}{q^2} + \frac{i\alpha_1}{2} - \frac{\lambda}{q^2} \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j)(q - q_j)^+ + \left(\frac{i}{2} + \frac{\lambda}{q}\right) \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j)\mathbb{1}_{q_j \le q}$$

Exercise 2.3. Find $\hat{q}$ for $J = 1$.

2.4. SEVERAL PRODUCTS WITH SCARCE RESOURCE

Suppose we manage the inventory of several products with a global constraint (budget, storage possibilities). We can clearly formulate the problem: Minimize

$$\begin{gathered}\sum_j C_j(q_j) \\ \sum_j a_j q_j \leq A\end{gathered},$$

with, whenever each individual cost is obtained by an EOQ formula

$$C_j(q_j) = \frac{K_j \lambda_j}{q_j} + h_j \frac{q_j}{2}.$$

We can introduce a Lagrange multiplier γ and obtain the system

$$\hat{q}_j(\gamma) = \sqrt{\frac{2K_j \lambda_j}{h_j + 2\gamma a_j}},$$

and $\gamma = \hat{\gamma} > 0$ is the unique solution of

$$\sum_j a_j \sqrt{\frac{2K_j \lambda_j}{h_j + 2\gamma a_j}} = A.$$

Note that if $A \geq \sum_j a_j \sqrt{\frac{2K_j \lambda_j}{h_j}}$, then the solution is simply that of the unconstrained problem (the constraint is not stringent). So we may assume

$$\sum_j a_j \sqrt{\frac{2K_j \lambda_j}{h_j}} > A,$$

and γ is uniquely defined.

We note a simple case in which γ can be computed explicitly. Suppose we have

$$h_j = i\alpha_j, a_j = k\alpha_j,$$

where α_j is the procurement price of product j. This situation occurs when h_j can be interpreted as an opportunity cost and i is an interest rate, and when the constraint is on the total value of the stocks.

Exercise 2.4. Show that

$$\hat{\gamma} = \frac{k}{2A^2} \left(\sum_j \sqrt{2K_j \lambda_j \alpha_j} \right)^2 - \frac{i}{2k}.$$

2.5. CONTINUOUS PRODUCTION OF SEVERAL PRODUCTS

We consider the situation of continuous production of several products using the same manufacturing system. Call r_j the production rate per unit of time of product j. We recall that $r_j > \lambda_j$. Since the manufacturing system is the same, we must have

$$\sum_j \frac{r_j}{\lambda_j} < 1.$$

We shall force a common cycle T for all products, so for instance we have the same number of orders $\frac{1}{T}$ per year. This may not be optimal for a long period, but it simplifies the analysis considerably.

The production (order) of product j is $q_j = T\lambda_j$. So we use T as the decision variable, and no more the level of orders. The EOQ formula for product j (expressed in T) leads to

$$C_j(T) = \frac{K_j}{T} + \frac{h_j}{2}\lambda_j T\left(1 - \frac{\lambda_j}{r_j}\right),$$

and we must minimize $\sum_j C_j(T)$.

Exercise 2.5. Show that the optimal $T = \hat{T}$ is

$$\hat{T} = \sqrt{\frac{2\sum_j K_j}{\sum_j h_j\lambda_j(1 - \frac{\lambda_j}{r_j})}}.$$

2.6. LEAD TIME

We have considered previously situations in which delivery is immediate, or is done at some rate per unit of time. The situation of Lead time is one when some delay is required to get the order (this delay is called the Lead Time). We will study the consequence of lead time on the EOQ model.

2.6.1. NO SHORTAGE ADMITTED. If L is the lead time, an order q will be delivered L units of time after the order is performed. If no shortage is admitted and if we do not perform any order during a lead time period then we must have the constraint

$$q \geq \lambda L,$$

otherwise we lose stock during a lead time period. After some time the stock will become negative which is not admitted. During a cycle (the period between two deliveries) of length $\frac{q}{\lambda}$ the average cost is still

$$C(q) = K\frac{\lambda}{q} + \frac{hq}{2}.$$

The EOQ problem becomes

$$\min_{q \geq \lambda L} C(q).$$

In particular if $\hat{q}$ defined by the EOQ formula satisfies the constraint, then it is the optimal value, whatever the value of L is. However, the order is no more made when the inventory is 0 but when the inventory is λL. If $\hat{q}$ does not satisfy the constraint, then the optimal order is simply λL, since by convexity $C(q)$ will be increasing for $q \geq \lambda L$. In this situation, one orders at time of delivery, and the stock at this time is also λL.

2.6.2. POSSIBILITY OF BACKLOG. We now describe an inventory model with lead time and the possibility of backlogging, see [**41**]. Backlog means that a negative inventory is possible. By negative inventory, one refers to the fact that the demand which cannot be met is recorded. It can be met later, but naturally there is a penalty in this case. The demand is continuous with a rate per unit of time λ. The structure of cost is the following: there is a fixed ordering cost K and a variable ordering cost c (per unit of quantity ordered); a holding cost per unit of quantity h, and a backlogging penalty per unit of quantity backlogged p.

There is a lead time L, between the time of order and the delivery. There is no order, during a lead time period. We have seen that in the case without backlog, the stock at the time when an order is put is necessarily $s = \lambda L$. Here, since backlog is possible, this relation is not necessarily true. Therefore, s is a decision variable like the amount q ordered. So the present problem is an optimisation problem, with two variables s, q.

Since there is no order during the lead time period, the inventory, just before the delivery is $\nu = s - \lambda L$. Just after delivery, it is $s - \lambda L + q$. In order to avoid a pathological situation of accumulation of orders, we will impose the constraint

$$q - \lambda L > 0.$$

The lowest value of the inventory is ν. If there is backlog, this number is necessarily negative. It is convenient to take the pair ν, q as decision variables. The value of the inventory at the time of order is simply $s = \nu + q$.

A cycle is the period T between two deliveries. We compute the various costs (holding and backlogging) during a cycle, recalling that there is no new order before the delivery is accomplished. We can take the origin of time at a point of order. So the first delivery takes place at L, and the period covering a cycle is $(L, T + L)$. At time L the stock is $q + s - \lambda L = q + \nu$. It is then depleted at a rate λ till the next delivery, in which it becomes $s - \lambda L = \nu$.

So we have $q = \lambda T$. Note that the condition on q imposes $T > L$. The holding cost during a cycle is given by the formula

$$h \int_L^{T+L} (q + \nu - \lambda(t - L))^+ dt = h\frac{((q+\nu)^+)^2}{2\lambda} - h\frac{((\nu)^+)^2}{2\lambda}.$$

Similarly the backlogging cost is given by

$$p \int_L^{T+L} (q + \nu - \lambda(t - L))^- dt = -p\frac{((q+\nu)^-)^2}{2\lambda} + p\frac{((\nu)^-)^2}{2\lambda}.$$

The ordering cost is $K + cq$. The average cost is obtained by dividing by the duration of the cycle T.

Collecting results, we obtain the following average cost

$$\begin{aligned} C(\nu, q) = c\lambda + \frac{\lambda K}{q} + h\left[\frac{((q+\nu)^+)^2}{2q} - \frac{(\nu^+)^2}{2q}\right] \\ + p\left[\frac{(\nu^-)^2}{2q} - \frac{((q+\nu)^-)^2}{2q}\right], \end{aligned}$$

with the constraint $q > \lambda L$. The function $C(\nu, q)$ is clearly increasing in ν, on $[0, +\infty)$, so we may assume $\nu \leq 0$.

Therefore $C(\nu, q)$ is given by

$$C(\nu, q) = c\lambda + \frac{\lambda K}{q} + h\frac{((q+\nu)^+)^2}{2q} + p\left[\frac{\nu^2}{2q} - \frac{((q+\nu)^-)^2}{2q}\right].$$

For $\nu \leq -q$, the function is decreasing in ν. Its smallest value is attained at $\nu = -q$, for which

$$c(-q, q) = c\lambda + \frac{\lambda K}{q} + p\frac{q}{2}.$$

For $\nu + q \geq 0$ we have

$$C(\nu, q) = c\lambda + \frac{\lambda K}{q} + h\frac{(q+\nu)^2}{2q} + p\frac{\nu^2}{2q}.$$

This function attains its minimum in ν at

$$\nu(q) = -(1-\varpi)q, \; \varpi = \frac{p}{h+p}.$$

One can see that a certain level of backlog is acceptable. If $p = \infty$, we get $\nu(q) = 0$. This makes sense because the penalty for backlog is infinite, so backlog is forbidden.

Concerning the cost we have

$$C(\nu(q), q) = c\lambda + \frac{\lambda K}{q} + [h\varpi^2 + p(1-\varpi)^2]\frac{q}{2}.$$

Noting that

$$h\varpi^2 + p(1-\varpi)^2 = \frac{hp}{h+p} = h\varpi < p,$$

we can assert that

$$\min_{\nu} C(\nu, q) = C(\nu(q), q) = c\lambda + \frac{\lambda K}{q} + h\varpi\frac{q}{2}.$$

It remains to minimize $C(\nu(q), q)$ in q, for $q > \lambda L$. The result is immediate

$$\hat{q} = \max\left\{\lambda L, \sqrt{\frac{2K\lambda}{h\varpi}}\right\},$$

and

$$\min_{\{\nu, q > \lambda L\}} C(\nu, q) = c\lambda + \begin{cases} \sqrt{2K\lambda h\varpi}, & \text{if } L \leq \sqrt{\frac{2K}{h\lambda\varpi}} \\ \frac{K}{L} + \frac{hL\lambda\varpi}{2}, & \text{if } L \geq \sqrt{\frac{2K}{h\lambda\varpi}} \end{cases}$$

When $L = 0$ and $p = \infty$, which implies $\varpi = 1$, we recover the EOQ formulas.

Remark. Consider any pair ν, q satisfying the conditions

$$\nu \leq 0, \; q \geq \lambda L, \; s = \nu + \lambda L > 0.$$

These conditions are satisfied, in particular, for a pair $\nu(q), q$, where $\nu(q)$ is given as before, and q is arbitrary. The cycle time length is T with $q = \lambda T$, and $T > L$. Let us consider, as above, the origin of time when an order is made, and a cycle starting at L. The During the cycle $L, T + L$, the inventory declines from $\nu + q$ to ν. It vanishes at time $L + T + \frac{\nu}{\lambda}$ and remains negative till $L + T$.

The average backlog is thus given by

$$\frac{1}{T}\int_{L+T+\frac{\nu}{\lambda}}^{L+T} \lambda\left[t - (L + T + \frac{\nu}{\lambda})\right] dt = \frac{\nu^2}{2T\lambda}.$$

During this backlog period, the waiting time till delivery is $L + T - t$, when t runs from $L + T + \frac{\nu}{\lambda}$ to $L + T$, hence the average waiting time is

$$\frac{1}{T}\int_{L+T+\frac{\nu}{\lambda}}^{L+T} (L + T - t)dt = \frac{\nu^2}{2T\lambda^2}.$$

So we have

$$\text{average backlog} = \lambda \text{ average waiting time}.$$

This is equivalent to the celebrated Little's Law, if we consider customers waiting in a queue. The backlog is analogous to the number of customers in the queue, and the waiting time to delivery is analogous to the waiting time of a customer.

2.7. RANDOM DEMAND RATE: UNSATISFIED DEMAND LOST

We turn now to a situation where the demand rate λ is random. The decision is defined as follows: when the stock is below $s > 0$, we order a quantity q. We assume that there is a lead-time L in the delivery, but because λ is random, we cannot use $\nu = s - \lambda L$. So we take as decision variables the pair s, q. Unlike the previous deterministic backlog model, we will assume that a demand which is not met is lost. This is similar to the Newsvendor model. Naturally, there is a penalty when this occurs. If fact, this assumption will simplify considerably the treatment. To understand the reason, we turn to the cycle. Define the cycle as the length of time between two situations when the inventory is s. Let us still denote by T this length of time. It is now a random variable. So to verify the constraint $T > L$ is not clear any more. We shall check that this possible when we assume $q > s$, thanks to our assumption that no backlog is permitted.

The stock, just before the time of delivery, is $(s - \lambda L)^+$ and $q + (s - \lambda L)^+$, just after. If $T > L$, during an interval of time of length $T - L$ the demand is $\lambda(T - L)$ and the drop of the inventory is $q + (s - \lambda L)^+ - s$. It follows that

$$\lambda T = q + (s - \lambda L)^- > \lambda L,$$

from the condition satisfied by s. The problem is to minimize the expected average cost denoted by $C(s, q)$.

Exercise 2.6. Check the formula

$$C(s,q) = E\frac{1}{2(q + (s - \lambda L)^-)}[h(q^2 + 2q(s - L\lambda)^+) + p((s - L\lambda)^-)^2 + 2\lambda(K + cq)].$$

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CHAPTER 3

MARKOV CHAINS

We present in this chapter the main results on Markov Chains, which are used to model state evolutions. There is no control of the evolution, in this chapter.

3.1. NOTATION

Let X be a metric space. We will limit ourselves to either R^d or a finite or countable set. X is called the *state space*. We consider the Borel $\sigma-$algebra $\mathcal{X}$ if $X = R^d$ or the $\sigma-$algebra of all subsets of X if X is finite or countable. A transition probability $\pi(x,\Gamma)$ is defined on $X \times \mathcal{X}$ with the following conditions

$$\begin{aligned}&\forall x, \pi(x,.) \text{ is a probability on } \mathcal{X},\\ &\forall \Gamma \text{ fixed } \pi(x,\Gamma) \text{ is a Borel function.}\end{aligned} \tag{3.1.1}$$

We define

$$\Omega = X^N,$$

where N represents the set of positive integers. So an element of Ω is written as

$$\omega = (\omega_1, \cdots, \omega_n, \cdots).$$

The canonical stochastic process (in discrete time) is defined by

$$y_n(\omega) = \omega_n.$$

We associate with Ω the $\sigma-$algebra $\mathcal{A} = \mathcal{X}^N$ generated by the rectangles

$$A_1 \times \cdots A_n, \forall n$$

$$A_1 \cdots A_n \in \mathcal{X}.$$

Given a probability distribution m on $\mathcal{X}$ which will be the probability distribution of the initial state y_1 we will define the probability distribution of the trajectory y_n associated to m, π.

The probability law of the trajectory will originate from a probability P on $\Omega, \mathcal{A}$ such that, for any test function $\phi(x_1, \cdots, x_n)$ and any n we have

$$\begin{aligned}&E\phi(y_1, \cdots, y_n)\\ &= \int dm(x_1) \int \pi(x_1, dx_2) \int \cdots \int \pi(x_{n-1}, dx_n)\phi(x_1, \cdots, x_n).\end{aligned} \tag{3.1.2}$$

One can show that it is possible to define P such that relation (3.1.2) holds (Kolmogorov theory, see for instance, M. Loeve [**31**]).

3.2. CHAPMAN-KOLMOGOROV EQUATIONS

3.2.1. FUNCTIONAL SET UP. Let B be the Banach space of measurable functions of X to R which are bounded, equipped with the norm

$$||f|| = \sup_x |f(x)|.$$

We define the operator $\Phi \in \mathcal{L}(B; B)$ by the formula

$$\Phi f(x) = \int_X f(\eta)\pi(x, d\eta).$$

Clearly one has

$$||\Phi|| \leq 1$$

Exercise 3.1. Show that

$$E[f(y_{n+1})|\mathcal{Y}^n] = \Phi f(y_n), \tag{3.2.1}$$

where

$$\mathcal{Y}^n = \sigma(y_1, \cdots, y_n).$$

Exercise 3.2. Show that

$$E[f(y_{n+1})|y_1] = \Phi^n f(y_1). \tag{3.2.2}$$

When $m = \delta_x$ we use the notation

$$E[f(y_{n+1})|y_1 = x] = f(y_{n+1}) = \Phi^n f(x) = \int P(x, n, d\eta) f(\eta),$$

and $P(x, n, d\eta)$ represents the probability distribution of y_n given $y_1 = x$.

Let Γ be a Borel subset of X, we write

$$P(x, n, \Gamma) = \int P(x, n, d\eta) 1\!\!1_\Gamma(\eta),$$

then we have the Chapman-Kolmogorov equation

$$P(x, n+m, \Gamma) = \int P(x, n, d\eta) P(\eta, m, \Gamma),$$

and also

$$E[1\!\!1_\Gamma(y_{k+n})|\mathcal{Y}^k] = P(y_k, n, \Gamma). \tag{3.2.3}$$

3.2.2. FUNCTIONS WITH LINEAR GROWTH. We consider here that $X = R^d$. It is important for the applications to Inventory Control to extend the operator Φ to functions which are not just bounded, but have linear growth. Let B_1 be the Banach space of measurable functions from R^d to R which have linear growth, equipped with the norm

$$||f||_1 = \sup_x \left| \frac{f(x)}{1 + |x|} \right|.$$

The operator Φ is not automatically defined on B_1. Suppose first that f is positive, then one can consider the sequence $f_M = f \wedge M$ and one can consider the sequence Φf_M. It is an increasing sequence. So it has a limit possibly ∞. We shall assume that

$$\frac{\Phi f_M(x)}{1 + |x|} \text{ bounded,}$$

so the limit is finite and defines $\Phi f(x)$. When f is not positive, we write $f = f^+ - f^-$ and by linearity

$$\Phi f(x) = \Phi f^+(x) - \Phi^- f(x).$$

The operator $\Phi \in \mathcal{L}(B_1; B_1)$ but we do not have the property $|\Phi||_1 \leq 1$. That complicates matters, as we shall see.

3.3. STOPPING TIMES

Let ν be a stopping time with respect to $\mathcal{Y}^n$. We thus have, by definition

$$\{\nu \leq k\} \subset \mathcal{Y}^k, \forall k.$$

We recall the definition of the $\sigma-$algebra $\mathcal{Y}^\nu$

$$A \subset \mathcal{Y}^\nu \Leftrightarrow A \cap \{\nu \leq k\} \subset \mathcal{Y}^k, \forall k.$$

We then state

Lemma 3.1. *We have the relation*

$$E[1\!\!1_\Gamma(y_{\nu+n})|\mathcal{Y}^\nu] = P(y_\nu, n, \Gamma). \tag{3.3.1}$$

Exercise 3.3. Prove relation (3.3.1).

Among the stopping times, we shall consider exit times

$$\tau = \inf\{n : y_n \notin \mathcal{O}\},$$

$$\mathcal{O}\text{: Borel subset of } X.$$

3.4. SOLUTION OF ANALYTIC PROBLEMS

3.4.1. CAUCHY PROBLEM. We are interested in giving the probabilistic interpretation of some analytic problems. We begin with the Cauchy problem. Let $f, g \in B$. We consider the sequence of functions $\in B$ defined by the relations

$$\begin{aligned} u^N &= g, \\ u^n &= f + \Phi u^{n+1}, \end{aligned} \tag{3.4.1}$$

then we have the interpretation

$$u^1(x) = E\left[\sum_{n=1}^{N-1} f(y_n) + g(y_N)|y_1 = x\right]. \tag{3.4.2}$$

Exercise 3.4. Prove relation (3.4.2).

We next consider the functional equation, for $\alpha < 1$

$$u = f + \alpha\Phi u, \tag{3.4.3}$$

with the interpretation

$$u(x) = E\left[\sum_{n=1}^{\infty} \alpha^{n-1} f(y_n)|y_1 = x\right]. \tag{3.4.4}$$

An important question is to solve equation (3.4.3) when f has linear growth. We can state the following result

Proposition 3.1. *Assume that*

$$\sum_{n=1}^{\infty}\alpha^{n-1}\Phi^{n-1}|f\,|(x)<\infty,\forall x, \tag{3.4.5}$$

then

$$u(x)=\sum_{n=1}^{\infty}\alpha^{n-1}\Phi^{n-1}f(x), \tag{3.4.6}$$

is the unique solution of (3.4.3) *such that*

$$\alpha^n\Phi^n u(x)\to 0, \textit{ as } n\to\infty. \tag{3.4.7}$$

Proof. It is enough to assume $f>0$. Then condition (3.4.5) is necessary and sufficient to assert that the function $u(x)$ defined by (3.4.6) is well defined. Let us show first that it is a solution. Consider the truncated series

$$u_n(x)=\sum_{j=1}^{n}\alpha^{j-1}\Phi^{j-1}f(x),$$

which satisfies the recursion

$$u_{n+1}(x)=f(x)+\alpha\Phi u_n(x),\quad u_1(x)=f(x),$$

then we have $u_n(x)\le u(x)$ and $u_n(x)$ is monotone increasing. If $u_*(x)$ denotes the limit, we note that by Lebesgue Theorem $\Phi u_n(x)$ converges also to $\Phi u_*(x)$. Therefore u_* is solution of (3.4.3). But $u_*(x)=u(x)$. Therefore u is a solution. Now for any solution, we have by iteration

$$u(x)=u_n(x)+\alpha^n\Phi^n u(x),$$

therefore if u is the limit of u_n, necessarily the condition (3.4.7) is satisfied. Conversely, any solution which satisfies the condition coincides with the limit of u_n. This completes the proof. □

3.4.2. DIRICHLET PROBLEM. Let now $\mathcal{O}$ to be a Borel subset of X. We note τ the exit time of y_n from $\mathcal{O}$. By definition

$$y_n\in\mathcal{O},\forall 1\le n<\tau;\ y_\tau\in X-\mathcal{O},$$

and $\tau=1$ if $y_1\in X-\mathcal{O}$.

We consider the problem

$$\begin{aligned}u(x)&=f(x)+\alpha\Phi u(x),\forall x\in\mathcal{O},\\ u(x)&=g(x),\forall x\in X-\mathcal{O},\end{aligned} \tag{3.4.8}$$

where $f,g\in B$.

Theorem 3.1. *Problem* (3.4.8) *has one and only one solution in* B*, with the following interpretation*

$$\begin{aligned}u(x)&=E\left[\sum_{n=1}^{\tau-1}\alpha^{n-1}f(y_n)+\alpha^{\tau-1}g(y_\tau)|y_1=x\right],\\ &\textit{where}\sum_{n=1}^{0}=0.\end{aligned} \tag{3.4.9}$$

Proof. The existence and uniqueness of the solution u follows from the fact that we look for a fixed point of the map $T:B\to B$ defined by

$$Tw=\left|\begin{array}{l}f+\alpha\Phi w,\ \text{in }\mathcal{O},\\ g \text{ in } X-\mathcal{O},\end{array}\right.$$

and T is a contraction since $\alpha < 1$. Next, we can write

$$u(y_n)\mathbb{1}_{n\leq\tau-1} = f(y_n)\mathbb{1}_{n\leq\tau-1} + \Phi u(y_n)\mathbb{1}_{n\leq\tau-1},$$

or

$$u(y_n)\mathbb{1}_{n\leq\tau-1} = f(y_n)\mathbb{1}_{n\leq\tau-1} + E[u(y_{n+1})|\mathcal{Y}^n]\mathbb{1}_{n\leq\tau-1}.$$

Moreover $\mathbb{1}_{n\leq\tau-1}$ is $\mathcal{Y}^n$ measurable, since

$$\mathbb{1}_{n\leq\tau-1} = 1 - \mathbb{1}_{\tau\leq n}.$$

Therefore we get

$$\alpha^{n-1}u(y_n)\mathbb{1}_{n\leq\tau-1} = \alpha^{n-1}f(y_n)\mathbb{1}_{n\leq\tau-1} + \alpha^n E[u(y_{n+1})\mathbb{1}_{n\leq\tau-1}|\mathcal{Y}^n].$$

Taking the expectation it follows

$$\begin{aligned}&\alpha^{n-1}E[u(y_n)\mathbb{1}_{n\leq\tau-1}|y_1 = x]\\ &= \alpha^{n-1}E[f(y_n)\mathbb{1}_{n\leq\tau-1}|y_1 = x] + \alpha^n E[u(y_{n+1})\mathbb{1}_{n\leq\tau-1}|y_1 = x].\end{aligned}$$

We add up these relations over n. It follows

$$E\left[\sum_{n=1}^{\tau-1}\alpha^{n-1}u(y_n)|y_1 = x\right] = E\left[\sum_{n=1}^{\tau-1}\alpha^{n-1}f(y_n)|y_1 = x\right] + E\left[\sum_{n=1}^{\tau-1}\alpha^{n}u(y_{n+1})|y_1 = x\right].$$

This means also

$$E\left[\sum_{n=1}^{\tau-1}\alpha^{n-1}u(y_n)|y_1 = x\right] = E\left[\sum_{n=1}^{\tau-1}\alpha^{n-1}f(y_n)|y_1 = x\right] + E\left[\sum_{n=2}^{\tau}\alpha^{n-1}u(y_n)|y_1 = x\right].$$

Canceling terms, formula (3.4.9) is obtained. □

3.4.3. MARTINGALE PROPERTIES. We state

Theorem 3.2. *For any function $u \in B$ we have the property*

$$u(y_n) + \sum_{j=0}^{n-1}(u - \Phi u)(y_j) \quad \text{is a } P, \mathcal{Y}^n \text{ martingale} \tag{3.4.10}$$

Proof. We have to check that, for $n \geq m$

$$E\left[u(y_n) - u(y_m) + \sum_{j=m}^{n-1}(u - \Phi u)(y_j)|\mathcal{Y}^m\right] = 0.$$

Recall that

$$E[u(y_{j+1})|\mathcal{Y}^j] = \Phi u(y_j).$$

From this the result follows easily.

Similarly one checks that

$$\alpha^n u(y_n) + \sum_{j=0}^{n-1}\alpha^j(u - \alpha\Phi u)(y_j),$$

is a $P, \mathcal{Y}^n$ martingale. □

3.5. ERGODIC THEORY

The objective of Ergodic Theory is to study the behavior of the Markov chain at infinity. If we think of of the Markov chain as a dynamic system, starting at time 1, in state $y_1 = x$, then we are interested by the possible limit of y_n. If the system were deterministic, it would be the ordinary limit of the sequence, if it exists. Since y_n is random, we think more in terms of the limit of the probability distribution of y_n, which means the limit of $P(x, n; \Gamma)$. We speak of ergodic behavior, if this limit, not only exists, but is also independent of x. The limit probability $m(\Gamma)$ if it exists is called invariant measure. Invariant means that if the initial state were distributed according to $m(\Gamma)$, then the probability distribution of y_n would remain the same $m(\Gamma)$ (invariant). It is a steady state. This steady state presents a strong interest in applications.

3.5.1. INVARIANT MEASURE. We assume that the transition probability satisfies

$$\pi(x; d\eta) = \varpi(x, \eta)\mu(d\eta), \tag{3.5.1}$$

where $\mu(d\eta)$ is a positive measure on X (not necessarily a probability). Moreover, there exists a Borel set X_0 such that $\mu(X_0) > 0$ and

$$\varpi(x, \eta) \geq \delta = \delta(X_0) > 0, \forall x \in X, \eta \in X_0. \tag{3.5.2}$$

A probability measure m on $X, \mathcal{A}$ is invariant with respect to the transition probability $\pi(x, \Gamma)$ whenever

$$m(\Gamma) = \int_X m(dx)\pi(x, \Gamma), \forall \Gamma. \tag{3.5.3}$$

It is easily seen that

$$m(\Gamma) = \int_X m(dx)P(x, n, \Gamma), \forall \Gamma. \tag{3.5.4}$$

The right-hand side represents the probability law of y_n when the law of y_1 is m. So the relation (3.5.4) shows that this probability is also m, which explains the word *invariant*.

3.5.2. THE MAIN RESULT. We present here a classical result, see J.L. Doob [**18**].

Theorem 3.3. *Under the assumptions* (3.5.1), (3.5.2), *there exists a unique invariant measure m with respect to the transition probability $\pi(x, \Gamma)$ and moreover*

$$\begin{gathered} \left\| \Phi^n f - \int_X f(x)m(dx) \right\| \leq 2||f||\beta^{n-1} \\ \forall f \in B, \ \text{with } 0 \leq \beta < 1, \ \text{independent of } f. \end{gathered} \tag{3.5.5}$$

We say that the process is ergodic, which means that its probability law $P(x, n, \Gamma)$ converges as n goes to ∞ to the probability law m, whatever the initial state $y_1 = x$ might be.

PROOF. Suppose $f \geq 0$. We define

$$m_n(f) = \inf_x \Phi^n f(x),$$
$$M_n(f) = \sup_x \Phi^n f(x).$$

Hence

$$\begin{aligned} m_n(f) &\le m_{n+1}(f),\\ M_n(f) &\ge M_{n+1}(f),\\ m_n(f) &\le M_n(f). \end{aligned}$$

We have successively

$$\begin{aligned} M_{n+1}(f) - m_{n+1}(f) &= \sup_x \Phi^{n+1} f(x) - \inf_x \Phi^{n+1} f(x)\\ &= \sup_{x,\xi} (\Phi^{n+1} f(x) - \Phi^{n+1} f(\xi))\\ &= \sup_{x,\xi} [\Phi(\Phi^n f)(x) - \Phi(\Phi^n f)(\xi)]\\ &= \sup_{x,\xi} \left[\int_X (\varpi(x,\eta) - \varpi(\xi,\eta)) \Phi^n f(\eta) \mu(d\eta) \right]. \end{aligned}$$

It follows

$$\begin{aligned} &M_{n+1}(f) - m_{n+1}(f)\\ &\quad\le \sup_{x,\xi} \Big[\int_X (\varpi(x,\eta) - \varpi(\xi,\eta))^+ \Phi^n f(\eta) \mu(d\eta)\\ &\qquad - \int_X (\varpi(x,\eta) - \varpi(\xi,\eta))^- \Phi^n f(\eta) \mu(d\eta) \Big]\\ &\quad\le \sup_{x,\xi} \Big[M_n(f) \int_X (\varpi(x,\eta) - \varpi(\xi,\eta))^+ \mu(d\eta)\\ &\qquad - m_n(f) \int_X (\varpi(x,\eta) - \varpi(\xi,\eta))^- \mu(d\eta) \Big]. \end{aligned}$$

Since

$$\int_X \varpi(x,\eta)\mu(d\eta) = \int_X \varpi(\xi,\eta)\mu(d\eta),$$

the coefficients of $M_n(f)$ and $m_n(f)$ are the same.

So we get

$$M_{n+1}(f) - m_{n+1}(f) \le (M_n(f) - m_n(f)) \sup_{x,\xi} \int_X (\varpi(x,\eta) - \varpi(\xi,\eta))^+ \mu(d\eta).$$

Next

$$\begin{aligned} &\int_X (\varpi(x,\eta) - \varpi(\xi,\eta))^+ \mu(d\eta)\\ &\quad= \int_X (\varpi(x,\eta) - \varpi(\xi,\eta)) \mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)\ge 0\}} \mu(d\eta)\\ &\quad= 1 - \int_X \varpi(x,\eta) \mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)< 0\}} \mu(d\eta)\\ &\qquad - \int_X \varpi(\xi,\eta)) \mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)\ge 0\}} \mu(d\eta)\\ &\quad\le 1 - \int_{X_0} \varpi(x,\eta) \mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)< 0\}} \mu(d\eta)\\ &\qquad - \int_{X_0} \varpi(\xi,\eta) \mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)\ge 0\}} \mu(d\eta), \end{aligned}$$

and from assumption (3.5.2) we deduce

$$\begin{aligned}&\leq 1-\delta\int_{X_0}\mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)<0\}}\mu(d\eta)\\&\quad-\delta\int_{X_0}\mathbb{1}_{\{\varpi(x,\eta)-\varpi(\xi,\eta)\geq 0\}}\mu(d\eta)\\&=1-\delta\mu(X_0).\end{aligned}$$

Therefore we have obtained

$$M_{n+1}(f)-m_{n+1}(f)\leq(M_n(f)-m_n(f))(1-\delta\mu(X_0)),$$

hence

$$M_n(f)-m_n(f)\leq(1-\delta\mu(X_0))^{n-1}||f||.$$

We deduce that

$$M_n(f)\Downarrow m(f);\quad m_n(m)\Uparrow m(f).$$

Also we have the inequality

$$M_n(f)-m_n(f)\geq\Phi^n f(x)-m(f)\geq m_n(f)-M_n(f).$$

We have obtained

$$|\Phi^n f(x)-m(f)|\leq M_n(f)-m_n(f)\leq(1-\delta\mu(X_0))^{n-1}||f||.$$

When f has no sign, using the decomposition of f between its positive et negative part, we obtain

$$|\Phi^n f(x)-m(f)|\leq M_n(f)-m_n(f)\leq 2(1-\delta\mu(X_0))^{n-1}||f||,$$

which is the desired result (3.5.5) with $\beta=1-\delta\mu(X_0)$.

It remains to prove that $m(f)$ defines an invariant measure, and that such an invariant measure is unique. The map $f\Rightarrow m(f)$ is linear and $|m(f)|\leq||f||$. If $f\geq 0$ then $m(f)\geq 0$, and $m(1)=1$. So $m(f)$ defines a probability on $X,\mathcal{X}$. Clearly

$$m(f)=m(\Phi f).$$

Applying with $f=\mathbb{1}_\Gamma$ yields

$$m(\Gamma)=\int_X m(dx)\pi(x,\Gamma),$$

which proves that m is an invariant measure. If $\tilde{m}$ is another invariant measure, we have by definition

$$\tilde{m}(f)=\tilde{m}(\Phi^n f)\Rightarrow m(f),$$

hence $\tilde{m}=m$. The proof has been completed. □

3.5.3. ANALYTIC PROBLEM.

Theorem 3.4. *Under the assumptions* (3.5.1), (3.5.2), *there exists one and only one pair $v\in B,\lambda\in R$ up to a constant for v such that*

$$\begin{aligned}&v-\Phi v+\lambda=f,\ \lambda=m(f)\\&m(v)=0,\\&m(v)-(m\Phi v)=0,\\&v(x)=\lim_{\alpha\to 1}\left(u_\alpha-\frac{\lambda}{1-\alpha}\right),\\&u_\alpha-\alpha\Phi u_\alpha=f,\alpha<1.\end{aligned}\tag{3.5.6}$$

PROOF. One has also

$$\lambda = \lim_{\alpha \to 1} (1-\alpha) u_\alpha, \tag{3.5.7}$$

and

$$\lambda = \lim \frac{1}{N} \sum_{j=1}^{N} \Phi^j f, \text{ in } B. \tag{3.5.8}$$

From equation (3.5.6) we deduce

$$m(v) - (m\Phi v) + \lambda = m(f). \tag{3.5.9}$$

Since m is invariant, we have $m(v) - (m\Phi v) = 0$, hence the second property (3.5.6). So λ is unique. If there are two solutions v_1, v_2, then one has

$$v_1 - v_2 = \Phi(v_1 - v_2).$$

So also

$$v_1 - v_2 = \Phi_n(v_1 - v_2) \Rightarrow m(v_1) - m(v_2).$$

So v_1 and v_2 differ only by a constant. If we impose the condition $m(v) = 0$, then the solution is unique. To prove existence we start with u_α defined in equation (3.5.6). Moreover

$$u_\alpha = \sum_{n=0} \alpha^n \Phi^n f. \tag{3.5.10}$$

So

$$u_\alpha - \frac{\lambda}{1-\alpha} = \sum_{n=0} \alpha^n \Phi^n (f - \lambda). \tag{3.5.11}$$

From property (3.5.5) we deduce that

$$||\Phi^n(f-\lambda)|| \leq 2||f||\beta^{n-1}.$$

It follows that the series defined in equation (3.5.11) is absolutely convergent in B. Moreover

$$u_\alpha - \frac{\lambda}{1-\alpha} \to v = \sum_{n=0}^{\infty} \Phi^n (f - \lambda).$$

Then v is the solution of equation (3.5.6). Finally property (3.5.8) follows from the fact that the series on the right hand side of (3.5.11) is absolutely convergent. □

3.6. EXAMPLES

3.6.1. INVENTORY WITH NO BACKLOG. We will consider the classical inventory problem with no backlog

$$y_{n+1} = (y_n + v_n - D_n)^+,$$

where D_n is the demand modeled as a sequence of independent identically distributed variables with density f. The quantity v_n is the order, for which we assume the following policy

$$v_n = S \mathbb{1}_{y_n = 0}.$$

So if we start with a stock which is less or equal to S, the stock remains always between 0 and S. So $X = [0, S]$. We easily see that this model defines a Markov chain on X equipped with the Borel $\sigma-$algebra, with transition probability

$$\pi(x, d\eta) = \begin{cases} \overline{F}(x)\delta(\eta) + \mathbb{1}_{0\leq\eta<x} f(x-\eta)d\eta, & \text{if } x > 0 \\ \overline{F}(S)\delta(\eta) + \mathbb{1}_{0\leq\eta<S} f(S-\eta)d\eta, & \text{if } x = 0 \end{cases} \tag{3.6.1}$$

So it is a mixture of a Dirac probability at 0 and probability density on positive inventories.

Assume $\overline{F}(S) > 0$ then the conditions of Theorem 3.3 are satisfied. We have

$$\mu(d\eta) = \delta(\eta) + d\eta,$$

and

$$\varpi(x, \eta) = \begin{cases} \overline{F}(x)\mathbb{1}_{\eta=0} + \mathbb{1}_{0<\eta<x} f(x-\eta), & \text{if } x > 0 \\ \overline{F}(S)\mathbb{1}_{\eta=0} + \mathbb{1}_{0<\eta<S} f(S-\eta), & \text{if } x = 0 \end{cases} \tag{3.6.2}$$

We can take $X_0 = 0$ and $\delta = \overline{F}(S)$ (this δ should not be confused with the Dirac measure $\delta(x)$).

We can in fact find the invariant measure directly. It is a probability measure $m(dx)$ on $[0, S]$, which we write as follows

$$\begin{aligned} m(dx) &= A\delta(x) + B(x)dx, \\ A + \int_0^S & B(x)dx = 1. \end{aligned} \tag{3.6.3}$$

To find A and $B(x)$ we must write the relation

$$\int_0^S \phi(x)m(dx) = \int_0^S \Phi\phi(x)m(dx), \forall\phi.$$

We obtain the system

$$\begin{aligned} A &= A\overline{F}(S) + \int_0^S \overline{F}(x)B(x)dx, \\ B(\eta) &= Af(S-\eta) + \int_\eta^S B(x)f(x-\eta)d\eta. \end{aligned} \tag{3.6.4}$$

then we have

Theorem 3.5. *Assume $f(x) > 0$ and continuous. The system* (3.6.4) *with the normalization condition* (3.6.3) *has a unique solution.*

PROOF. We note that

$$A = \frac{\int_0^S \overline{F}(x)B(x)dx}{F(S)}. \tag{3.6.5}$$

Replacing A in the second equation and taking $\eta = S$ yields

$$B(S) = f(0)\frac{\int_0^S \overline{F}(x)B(x)dx}{F(S)}.$$

Calling

$$u(\eta) = \frac{B(\eta)}{B(S)},$$

the second equation reduces to

$$u(\eta) = \frac{f(S-\eta)}{f(0)} + \int_\eta^S u(x) f(x-\eta)dx. \tag{3.6.6}$$

Then equation (3.6.6) has one and only one solution. Indeed the map

$$T : u \to \frac{f(S-\eta)}{f(0)} + \int_\eta^S u(x) f(x-\eta)dx,$$

is a contraction on $C([0, S])$, since

$$||Tu_1 - Tu_2|| \leq ||u_1 - u_2|| \int_0^S f(x)dx,$$

and

$$\int_0^S f(x)dx < 1,$$

and $B(S)$ is defined by

$$B(S) \left[\frac{\int_0^S \overline{F}(x)u(x)dx}{F(S)} + \int_0^S u(x)dx \right] = 1.$$

The proof has been completed. In the case of a Poisson distribution $f(x) = \lambda \exp -\lambda x$, one finds that $u(\eta) = 1$ and

$$B(\eta) = \frac{\lambda}{1 + \lambda S}; \quad A = \frac{1}{1 + \lambda S}.$$

□

3.6.2. INVENTORY WITH BACKLOG. The model is

$$y_{n+1} = y_n + v_n - D_n.$$

So negative inventories are permitted. The orders follow an s, S policy, with $s < S$, $S > 0$. Namely

$$v_n = 0, \text{ if } y_n > s,$$

$$v_n = S - y_n, \text{ if } y_n \leq s.$$

Clearly if $y_1 \leq S$ then $y_n \leq S, \forall n$.

We have

$$\begin{cases} \pi(x, d\eta) = f(S-\eta) 1\!\!1_{\eta \leq S} d\eta, & \text{if } x \leq s \\ \pi(x, d\eta) = 1\!\!1_{\eta \leq x} f(x-\eta)d\eta, & \text{if } x > s \end{cases} \tag{3.6.7}$$

We have $X = (-\infty, S]$ and $\mu(d\eta) = d\eta$, with

$$\varpi(x, \eta) = \begin{cases} 1\!\!1_{\eta \leq S} f(S-\eta), & \text{if } x \leq s \\ 1\!\!1_{\eta \leq x} f(x-\eta), & \text{if } x > s \end{cases}$$

Assume $f(x) > 0$ and continuous. Then the assumptions of Theorem 3.3 are satisfied. We take $X_0 = (s-1, s)$ and

$$\delta = \inf_{\{0 \leq x \leq S-s+1\}} f(x).$$

Again we can study directly the problem of existence and uniqueness of the invariant measure. We look for a density $m(x)$ with respect to Lebesgue measure on X. This density must satisfy the equation

$$m(\eta) = \int_{-\infty}^{S} m(x)\varpi(x,\eta)dx,$$

which amounts to

$$m(\eta) = f(S-\eta)\int_{-\infty}^{s} m(x)dx + \int_{s}^{S} m(x)f(x-\eta)\mathbb{1}_{\{\eta\leq x\}}dx$$

Exercise 3.5. Show that $m(\eta)$ is uniquely defined by the formulas

$$m(\eta) = \frac{f(S-\eta)+\int_s^S u(x)f(x-\eta)dx}{1+\int_s^S u(y)dy}, \text{ if } \eta \leq s$$
$$m(\eta) = \frac{u(\eta)}{1+\int_s^S u(y)dy}, \text{ if } s \leq \eta \leq S$$

where the function $u(\eta)$ is defined in s, S as the unique solution of the integral equation

$$u(\eta) = f(S-\eta) + \int_{\eta}^{S} u(x)f(x-\eta)dx, \quad s \leq \eta \leq S.$$

3.6.3. FINITE NUMBER OF STATES. Suppose X is composed of a finite number of states, denoted by $i = 1, \cdots, I$. The transition probability reduces to a matrix $\varpi(i,j)$. In order to apply Theorem 3.3 it is sufficient to assume that there exists a state j_0 such that

$$\varpi(i,j_0), \forall i \in X. \tag{3.6.8}$$

The unique invariant measure will satisfy

$$m_i = \sum_j m_j\varpi(j,i), \quad \sum_i m_i = 1, \quad m_i \geq 0$$

CHAPTER 4

OPTIMAL CONTROL IN DISCRETE TIME

A very comprehensive presentation of Discrete-Time Markov Control Processes can be found in the book of O. Hernández-Lerma and J.B. Lasserre, [**25**]. Our presentation discusses more completely the issue of uniqueness, and separates completely the study of the functional equation, resulting from Dynamic Programming (Bellman equation) from its interpretation in the context of optimal control.

4.1. DETERMINISTIC CASE

4.1.1. NOTATION. We begin with deterministic models, although the deterministic case can be embedded in the stochastic case. The reason is because in deterministic situations one does not have to worry about information. Decisions are just given values (open loop control). In the stochastic case, decisions are functions of the state of information (closed loop control). It is possible to reconcile the two approaches in the deterministic case, but this is unnecessarily complex just to obtain the major results.

We consider a dynamic system whose evolution is governed by

$$y_{n+1} = g(y_n, v_n), \quad y_1 = x. \tag{4.1.1}$$

At this stage, we do not need any specific assumptions on the function g, except measurability. As we shall see the first part of the discussion will be of algebraic nature. The state space is X, a metric space R^d or a discrete space. We next consider a function

$$l(x, v) \text{ bounded below.} \tag{4.1.2}$$

A control (or more precisely a control policy) is a sequence of values $v_1 \cdots v_n \cdots$ in a subset U of a metric space. A control is denoted by V. We define the payoff

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n). \tag{4.1.3}$$

Without loss of generality, we will assume $l(x, v) \geq 0$. Indeed If $l(x, v) + C \geq 0$, we can change $J_x(V)$ into $J_x(V) + \dfrac{C}{1-\alpha}$ and obtain the positivity assumption. The discount α is smaller than 1. We define

$$u(x) = \inf J_x(V), \tag{4.1.4}$$

which is a positive number. The function $u(x)$ is called the value function. We shall assume that

$$u(x) < \infty \tag{4.1.5}$$

4.1.2. BELLMAN EQUATION. The function $u(x)$ satisfies a functional equation, called Bellman equation, [**2**], [**11**]. As we shall see, it is simply the consequence of algebraic considerations. The study of this functional equation, independently of its origin, will be done later and requires analysis tools.

Proposition 4.1. *The function $u(x)$ satisfies the relation*

$$u(x) = \inf_{v \in U} [l(x, v) + \alpha u(g(x, v))]. \tag{4.1.6}$$

PROOF. From the definition, we can write

$$u(x) = \inf_{v_1} \left[l(x, v_1) + \alpha \inf_{v_2 \cdots v_n \cdots} \left[\sum_{n=2}^{\infty} \alpha^{n-2} l(y_n, v_n) \right] \right],$$

but the inf inside the bracket is clearly $u(g(x, v))$. Therefore the result follows. □

Suppose now that there exists a measurable function $\hat{v}(x)$ such that the infimum in the right hand side of (4.1.6) is attained for this value and for any value of x. Such a function is called a feedback. We can use this feedback in the state equation and define a sequence $\hat{y_n}$ as follows

$$\hat{y}_{n+1} = g(\hat{y}_n, \hat{v}(\hat{y}_n)).$$

Defining next $\hat{v_n} = \hat{v}(\hat{y}_n)$ we obtain a control $\hat{V} = (\hat{v}_1 \cdots \hat{v}_n \cdots)$. We then obtain the result

Proposition 4.2. *$\hat{V}$ is optimal.*

PROOF. By induction we can check that

$$u(x) = \sum_{j=1}^{n} \alpha^{n-1} l(\hat{y_n}, \hat{v_n}) + \alpha^n u(\hat{y}_{n+1}),$$

and thus $u(x) \geq J_x(\hat{V})$. However being the infimum, $u(x)$ is also smaller than $J_x(\hat{V})$. So we have equality and thus $\hat{V}$ is optimal. □

Note that the optimal control is obtained via a feedback on the state. This property is a curiosity in the deterministic case, but will play a fundamental role in the stochastic case, since it has an informational content.

4.1.3. EXTENSION WHEN THE SET OF CONSTRAINTS DEPENDS ON THE STATE. There is however a case when information matters, even in the deterministic case. It is the case when the set of constraints U is not fixed, but depends on the state $U(x)$. A control policy V cannot be a simple sequence v_n of elements of U, because U is not fixed anymore. We must also satisfy

$$v_n \in U(y_n).$$

However, it is not difficult to check that the argument of the proof of Proposition 4.1 remains valid. The Bellman equation is changed into

$$u(x) = \inf_{v \in U(x)} [l(x, v) + \alpha u(g(x, v))]. \tag{4.1.7}$$

4.2. STOCHASTIC CASE: GENERAL FORMULATION

We turn now to the stochastic case. As we shall see the machinery is more complex.

4.2.1. NOTATION AND ASSUMPTIONS. The probabilistic set up is the same as in Chapter 3. The state space is X, $\mathcal{X}$ and the probability space is $\Omega = X^N$, with $\sigma-$ algebra $\mathcal{A} = \mathcal{X}^N$. The canonical process is denoted by $y_n(\omega) = \omega_n$ and it generates the filtration $\mathcal{Y}^n$. We now consider a family of transition probabilities, indexed by a parameter v, the control variable. The parameter v belongs to a set U, the set of controls, which is a closed subset of a metric space.

The transition probability is denoted $\pi(x, v, d\eta)$. We associate to it the operator on B (space of measurable bounded functions on X), also indexed by v

$$\Phi^v f(x) = \int_X f(\eta)\pi(x, v, d\eta).$$

We now define as *a control policy* a sequence of Borel maps with values in U

$$v_1(\eta_1),\ v_2(\eta_1, \eta_2)$$

$$\cdots, v_n(\eta_1, \cdots, \eta_n), \cdots$$

Such a control is denoted by V. However, we see the main difference with the deterministic case, in which a control was simply a sequence of values. In the stochastic case, the control must be defined in a closed loop manner.

To a control V and an initial state x we associate a probability law on Ω, $\mathcal{A}$ which is the conditional law of the canonical process given $y_1 = x$. We denote it by $P^{V,x}$. For any test function $f(x_1, \cdots, x_n)$ we have

$$\begin{aligned} E^{V,x} f(y_1, \cdots, y_n) &= E[f(y_1, \cdots, y_n)|y_1 = x] \\ &= \int \pi(x, v_1(x), dx_2) \int \pi(x_2, v_2(x, x_2), dx_3) \cdots \\ &\quad \int \pi(x_{n-1}, v_{n-1}(x, x_2, \cdots, x_{n-1}), dx_n) f(x_1, \cdots, x_n). \end{aligned}$$

$P^{V,x}$ is a controlled Markov chain. A controlled Markov chain is not a Markov chain strictly speaking, since we keep memory of all previous states, because of the decision rule. However some analogy with the standard Markov chains can be made. We introduce

$$P^{V,x}(n-1; x, \cdots, x_{n-1}; n, \Gamma) = \pi(x_{n-1}, v_{n-1}(x, x_2, \cdots, x_{n-1}), \Gamma)$$

$$\begin{aligned} P^{V,x}(m; x, \cdots, x_m; n, \Gamma) &= \int \pi(x_m, v_m(x, x_2, \cdots, x_m), dx_{m+1} \cdots \\ &\int \pi(x_{n-2}, v_{n-2}(x, x_2, \cdots, x_{n-2}), dx_{n-1} \pi(x_{n-1}, v_{n-1}(x, x_2, \cdots, x_{n-1}), \Gamma) \end{aligned}$$

Exercise 4.1. Check that

$$P^{V,x}(k; y_1, \cdots, y_k; k+n, \Gamma) = E^{V,x}[\mathbb{1}_\Gamma(y_{k+n})|\mathcal{Y}^k],$$

which can be extended to replacing k by a stopping time ν. Check that

$$P^{V,x}(\nu; y_1, \cdots, y_\nu; \nu+n, \Gamma) = E^{V,x}[\mathbb{1}_\Gamma(y_{\nu+n})|\mathcal{Y}^\nu]$$

Check that for any $u \in B$

$$u(y_n) + \sum_{j=1}^{n-1} (u(y_j) - \Phi^{v_j(y_1, \cdots, y_j)} u(y_j)),$$

is a $\mathcal{Y}^n, P^{V,x}$ martingale. Check also that

$$\alpha^n u(y_n) + \sum_{j=1}^{n-1} \alpha^j (u(y_j) - \alpha \Phi^{v_j(y_1,\cdots,y_j)} u(y_j)),$$

is a $\mathcal{Y}^n, P^{V,x}$ martingale.

Remark. We can consider the deterministic case as a particular case of a Markov chain, in which

$$\pi(x, v, d\eta) = \delta_{g(x,v)}$$

the Dirac measure at the point $g(x, v)$. However the control must remain defined in a closed loop manner. We cannot recover the open loop control in this manner. This is why to go through this path to consider the deterministic problem is unnecessary complicated.

4.2.2. SETTING OF THE PROBLEM. Consider

$$l(x, v) \geq 0, \tag{4.2.1}$$

and U is a subset of a metric space.

We consider a controlled Markov chain described by its transition probability $\pi(x, v, \Gamma)$. Consider a control policy$V$ and construct the corresponding probability $P^{V,x}$ for the canonical process y_n. The functions $v_n(x_1, \cdots, x_n)$ appearing in the control must take values in U.

We define the cost functional

$$J_x(V) = E^{V,x} \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n). \tag{4.2.2}$$

If we accept the value $+\infty$ this quantity is well defined. The value function is defined by

$$u(x) = \inf_V J_x(V).$$

An important question is whether the value function satisfies a functional equation, as in the deterministic case, following algebraic arguments. This becomes highly non trivial, but we are going to show that it is possible, in a case which is useful for applications and mimics to some extent the deterministic case. In the general situation described above, we will avoid the difficulty by using arguments from analysis and not from algebra.

4.2.3. DIRECT APPROACH. We consider the situation when the evolution of the dynamic system is described as follows. Consider a probability space $(\Omega, \mathcal{A}, P)$, on which is defined a stochastic process $D_1, \cdots D_n \cdots$ made of *independent, identically distributed* random variables. The notation D recalls the demand process, which will appear constantly in the application to inventory control. We denote by $h(\xi)$ the probability density of a generic D, which lies in R^k to fix ideas. The process y_n is defined by

$$y_{n+1} = g(y_n, v_n, D_n) \quad y_1 = x.$$

We shall assume that the demand is observable. At time $n + 1$, we have observed the values of $D_1 \cdots D_n$. So

$$v_n = v_n(D_1, \cdots, D_{n-1}) \; n \geq 2,$$

and v_1 is deterministic. This is more general than $v_n = v_n(y_1, \cdots, y_n)$ and follows naturally from the information which is available at each time. If U depends on x, then we have the constraint

$$v_n(D_1, \cdots, D_{n-1}) \in U(y_n),$$

and y_n is also a function of $D_1, \cdots, D_{n-1}$. The operator $\Phi^v f(x)$ is given by

$$\Phi^v f(x) = Ef(g(x, v, D)) = \int f(g(x, v, \eta))\, d\eta. \tag{4.2.3}$$

Let $V = \{v_1, \cdots v_n \cdots\}$ be a control as defined above. The payoff is given by

$$J_x(V) = E \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n). \tag{4.2.4}$$

Note that the expectation is taken with respect to the fixed probability P and does not depend on the control policy as in formula (4.2.2). We define the value function

$$u(x) = \inf_V J_x(V).$$

Proposition 4.3. *The value function satisfies the functional equation (Bellman equation)*

$$u(x) = \inf_{v \in U(x)} [l(x, v) + \alpha E u(g(x, v, D))]. \tag{4.2.5}$$

PROOF. Let us set

$$K(x, v_1, \cdots, v_n, \cdots; D_1, \cdots, D_n, \cdots) = \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n),$$

which is a functional on $D_1, \cdots, D_n \cdots$ Thanks to the independence of the random variables, we have

$$\begin{aligned} &EK(x, v_1, \cdots, v_n, \cdots; D_1, \cdots, D_n, \cdots) \\ &\quad = E_{D_1 D_2 \cdots} K(x, v_1, \cdots, v_n, \cdots; D_1, \cdots, D_n, \cdots), \end{aligned}$$

in which the expectations as indices in the right hand side mean that one must take the expectations successively. We start backwards. The fact that there is an infinite number of expectations is not a problem, since each term in the expression of K depends on a finite number of demands. Note the relation

$$\begin{aligned} &K(x, v_1, \cdots, v_n, \cdots; D_1, \cdots, D_n, \cdots) \\ &\quad = l(x, v_1) + \alpha K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots). \end{aligned}$$

Therefore, for fixed x, v_1, D_1 we can write

$$\inf_{v_2 \cdots, v_n, \cdots} E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) = u(g(x, v_1, D_1)). \tag{4.2.6}$$

Now the key point is to check that

$$\begin{aligned} &E_{D_1} \inf_{v_2 \cdots, v_n, \cdots} E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) \\ &\quad = \inf_{v_2 \cdots, v_n, \cdots} E_{D_1 D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots). \end{aligned} \tag{4.2.7}$$

In other words, we can exchange the two operators E_{D_1} and inf. The reason is because the decision variables $v_2, \cdots, v_n, \cdots$ are functions of all the random variables, including D_1. More specifically, we can first assert that

$$\begin{aligned} &E_{D_1} \inf_{v_2 \cdots, v_n, \cdots} E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) \\ &\quad \leq E_{D_1 D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) \end{aligned}$$

where, in the right hand side, the decision variables are arbitrary. Therefore the left hand side of equation (4.2.7) is smaller than its right hand side. On the other hand for any ϵ and any fixed D_1 we can find a policy $v_2^\epsilon, \cdots, v_n^\epsilon, \cdots$ depending of course of D_1 such that

$$\begin{aligned} &E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2^\epsilon \cdots, v_n^\epsilon, \cdots; D_2, \cdots, D_n, \cdots) \\ &\quad \leq \inf_{v_2 \cdots, v_n, \cdots} E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) + \epsilon. \end{aligned}$$

Therefore

$$\begin{aligned} &E_{D_1 D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2^\epsilon \cdots, v_n^\epsilon, \cdots; D_2, \cdots, D_n, \cdots) \\ &\quad \leq E_{D_1} \inf_{v_2 \cdots, v_n, \cdots} E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) + \epsilon. \end{aligned}$$

Therefore also

$$\begin{aligned} &\inf_{v_2 \cdots, v_n, \cdots} E_{D_1 D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) \\ &\quad \leq E_{D_1} \inf_{v_2 \cdots, v_n, \cdots} E_{D_2 D_3 \cdots} K(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots) + \epsilon, \end{aligned}$$

and since ϵ is arbitrary, the right hand side of (4.2.7) is smaller than the left hand side. Hence the relation (4.2.7) holds. This relation means also that

$$Eu(g(x, v_1, D_1)) = \inf_{v_2 \cdots v_n \cdots} EK(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots).$$

But then

$$u(x) = \inf_{v_1} [l(x, v_1) + \alpha \inf_{v_2 \cdots v_n \cdots} EK(g(x, v_1, D_1), v_2 \cdots, v_n, \cdots; D_2, \cdots, D_n, \cdots)],$$

which implies the result. □

4.3. FUNCTIONAL EQUATION

We begin with assumptions. The space $X = R^d$. The set of controls U is a closed subset of R^k. If $\phi(x, v)$ is a function of two arguments, we write $\phi_v(x) = \phi(x, v)$. We shall assume that

$$\begin{aligned} &\Phi^v \phi_v(x) \text{ is uniformly continuous in } v, x \text{ if} \\ &\phi_v(x) = \phi(x, v) \text{ is uniformly continuous and bounded in } x, v. \end{aligned} \tag{4.3.1}$$

We also assume

$$\begin{aligned} &l_v(x) = l(x, v) \text{ l.s.c } \geq 0; \\ &l(x, v) \text{ is bounded on bounded sets;} \\ &\Phi^v l_v(x) \text{ is bounded on bounded sets;} \\ &\{v \in U | l(x, v) \leq L, |x| \leq M\} \subset \{|v| \leq K_{LM}\}. \end{aligned} \tag{4.3.2}$$

4.3.1. TECHNICAL RESULTS. We begin by recalling a few important technical results. Some proofs are in the Appendix.

Lemma 4.1. *Let f be a map from a metric space X into $(-\infty, +\infty]$, which is l.s.c. and bounded below. There exists a sequence $f_n(x) \in C(X)$ (space of uniformly continuous bounded functions on X) such that $f_n(x) \uparrow f(x)$ point wise.*

Lemma 4.2. *Under the assumption* (4.3.1) *if $l_v(x) = l(x, v)$ is l.s.c. and bounded below then the function $\Phi^v l_v(x)$ of the pair v, x is l.s.c. and bounded below.*

Lemma 4.3. *Let $F(x, v)$ be l.s.c. in both arguments and bounded below. If U is metric compact then*

$$G(x) = \inf_{v \in U} F(x, v),$$

is also l.s.c. and bounded below.

A key result is the existence for each x of a minimum which is a measurable function of x.

Theorem 4.1. *Under the assumptions of Lemma 4.3, there exists a Borel function $\hat{v}(x) : X \to U$ such that*

$$G(x) = F(x, \hat{v}(x)), \forall x.$$

The fact that U is compact may not be verified in some applications. We present here a situation which occurs in applications and where the compactness assumption of U is removed. Let us assume that

$$\begin{aligned} &U \text{ is a closed subset of } R^d; \\ &F(x,v) \text{ is l.s.c and bounded below;} \\ &\{v | F(x, v) \leq F(x, v_0)\} \subset \{v | \, |v| \leq \gamma(x)\}; \\ &\gamma(x) \text{ is bounded on bounded sets.} \end{aligned} \tag{4.3.3}$$

We state the

Theorem 4.2. *Under the assumptions of Lemma 4.3 except U compact, and if the assumption* (4.3.3) *holds, then $G(x)$ is l.s.c. and bounded below, and there exists a Borel map $\hat{v}(x)$ which achieves the minimum for any x.*

4.3.2. CEILING FUNCTION. We consider first the case of a fixed control $v_0 \in U$ as a parameter. Introduce the equation

$$w_0(x) = l(x, v_0) + \alpha \Phi^{v_0} w(x). \tag{4.3.4}$$

More precisely we consider the function

$$w_0(x) = \sum_{n=1}^{\infty} \alpha^{n-1} (\Phi^{v_0})^{n-1} l_{v_0}(x), \tag{4.3.5}$$

where we have noted $l_v(x) = l(x, v)$.

The function $w_0(x)$ is defined as the limit as $N \uparrow +\infty$ of

$$w_{0N}(x, v) = \sum_{n=1}^{N} \alpha^{n-1} (\Phi^{v_0})^{n-1} l_{v_0}(x).$$

Since all the functions are positive, the limit is well defined, but may take the value $+\infty$. Recalling Lemma 4.2, we can assert that the function $w_{0N}(x,v)$ is l.s.c. and therefore the function $w(x,v)$ is also l.s.c. We will use in the sequel the assumption

$$w_0(x) < \infty, \forall x. \tag{4.3.6}$$

$$w_0(x) \text{ is bounded on bounded sets}$$

4.3.3. BELLMAN EQUATION.

We consider the functional equation

$$u(x) = \inf_{v \in U}[l(x,v) + \alpha \Phi^v u(x)], \forall x. \tag{4.3.7}$$

This equation is now considered, without any interpretation as the value function of a control problem. We state the main result

Theorem 4.3. *Under the assumptions* (4.3.1), (4.3.2), (4.3.6) *the set of solutions of the functional equation* (4.3.7) *satisfying*

$$0 \leq u(x) \leq w_0(x) \forall x,$$

is not empty and has a minimum and a maximum solution, denoted respectively $\underline{u}(x)$ *and* $\overline{u}(x)$. *The minimum solution is l.s.c.*

Proof. Consider the following iteration

$$u_0(x) = 0,$$
$$u_{n+1}(x) = \inf_{v \in U}[l(x,v) + \alpha \Phi^v u_n(x)].$$

This sequence is monotone increasing. We have

$$0 \leq u_n(x) \leq w_0(x) \; \forall x,$$

which can be checked recursively.

We thus have $u_n(x) \uparrow u(x)$ hence

$$u_{n+1}(x) \leq \inf[l(x,v) + \alpha \Phi^v u(x)], \forall x,$$

and thus also

$$u(x) \leq \inf[l(x,v) + \alpha \Phi^v u(x)], \forall x.$$

Moreover, thanks to the assumptions (4.3.2) the functions $u_n(x)$ are l.s.c. so there exists a Borel map $v_n(x)$ such that

$$u_{n+1}(x) = l(x, v_n(x)) + \alpha \Phi^{v_n(x)} u_n(x).$$

Let $n \leq m$ we can write

$$\begin{aligned} u_{m+1}(x) &= l(x, v_m(x)) + \alpha \Phi^{v_m(x)} u_m(x) \\ &\geq l(x, v_m(x)) + \alpha \Phi^{v_m(x)} u_n(x), \end{aligned}$$

hence also

$$u(x) \geq l(x, v_m(x)) + \alpha \Phi^{v_m(x)} u_n(x).$$

Now for x fixed, the sequence $v_m(x)$ remains in a compact set and thus there exists a subsequence such that $v_{m_k}(x) \to v^*(x)$. Remember that n is fixed and that $f(x,v) + \alpha \Phi^v u_n(x)$ is l.s.c.

We thus can state

$$u(x) \geq l(x, v^*(x)) + \alpha \Phi^{v^*}(x) u_n(x).$$

Letting now n go to ∞ we obtain

$$u(x) \geq l(x, v^*(x)) + \alpha \Phi^{v^*}(x) u(x),$$

hence

$$u(x) \geq \inf_v [l(x,v) + \alpha \Phi^v u(x)].$$

Since the reverse inequality is true, the function u is a solution. This function is the smallest solution, since if we have another solution $\tilde{u}$ such that $0 \leq \tilde{u} \leq w_0(x)$ then we check sequentially that $\tilde{u} \geq u_n$ hence in the limit $\tilde{u} \geq u$. We denote the minimum solution by $\underline{u}(x)$. This minimum solution is l.s.c. We turn now to the maximum solution. We define a decreasing sequence by starting with the ceiling function and using the same iteration, namely

$$\begin{aligned} &u^0(x) = w_0(x), \\ &u^{n+1}(x) = \inf_{v \in U} [l(x,v) + \alpha \Phi^v u^n(x)]. \end{aligned}$$

We have $u^n(x) \downarrow u(x)$ hence also

$$u^{n+1}(x) \geq \inf_v [l(x,v) + \alpha \Phi^v u(x)].$$

Therefore,

$$u(x) \geq \inf_v [l(x,v) + \alpha \Phi^v u(x)].$$

On the other hand, for any v we can write the inequality

$$u^{n+1}(x) \leq l(x,v) + \alpha \Phi^v u^n(x).$$

To pass to the limit we have to be careful that $u^n(x)$ is decreasing. We cannot use Fatou's Lemma directly. However the sequence $w_0(x) - u^n(x)$ is increasing and positive. Therefore, applying Fatou's Lemma to this sequence, we can assert than

$$\Phi^v u^n(x) \downarrow \Phi^v u(x).$$

It follows that

$$u(x) \leq \inf_v [l(x,v) + \alpha \Phi^v u(x)],$$

and since the reverse inequality is true, the function u is a solution of equation (4.3.7). One can check easily that it is the maximum solution, within the interval $0, w_0(x)$ This maximum solution is denoted by $\overline{u}(x)$. The proof has been completed. □

4.4. PROBABILISTIC INTERPRETATION

4.4.1. THE MINIMUM SOLUTION. We recall the monotone increasing process

$$\begin{aligned} &u_0(x) = 0, \\ &u_{n+1}(x) = \inf_{v \in U} [l(x,v) + \alpha \Phi^v u_n(x)]. \end{aligned} \tag{4.4.1}$$

The function $u_n(x)$ is l.s.c. and bounded below. There will exist a Borel function $\hat{v}_n(x)$ with values in U such that

$$u_{n+1}(x) = l(x, \hat{v}_n(x)) + \alpha \Phi^{\hat{v}_n(x)} u_n(x).$$

Consider the cost function defined by (4.2.2). Let us define the truncated cost function

$$J_x^n(V) = E^{V,x} \left[\sum_{n=1}^{n} \alpha^{n-1} l(y_n, v_n) \right].$$

We define from the feedback introduced above, the control policy

$$\hat{V}_n = (\hat{v}_{1n}, \cdots, \hat{v}_{jn}, \cdots),$$

where

$$\hat{v}_{jn}(x_1,\cdots,x_j)=\hat{v}_j(x_j),\forall 1\le j\le n,$$
$$\hat{v}_{jn}(x_1,\cdots,x_j)=\bar{v}\text{ (arbitrary) }\forall j\ge n+1.$$

We can state

Lemma 4.4. *We have the interpretation*

$$u_n(x)=\inf_V J_x^n(V) \tag{4.4.2}$$

PROOF. Pick any control V. For $j=0,\cdots,n-1$ we can write

$$\begin{aligned} u_{j+1}(y_{n-j}) &\le l(y_{n-j},v_{n-j})+\alpha\Phi^{v_{n-j}}u_j(y_{n-j}) \\ &= l(y_{n-j},v_{n-j})+\alpha E^{V,x}[u_j(y_{n-j+1})|\mathcal{Y}^{n-j}]. \end{aligned}$$

Hence also

$$E^{V,x}\alpha^{n-j-1}u_{j+1}(y_{n-j})\le E^{V,x}\alpha^{n-j-1}l(y_{n-j},v_{n-j})+E^{V,x}\alpha^{n-j}u_j(y_{n-j+1}).$$

Adding up from $j=0,\cdots,n-1$ we get

$$u_n(x)\le J_x^n(V).$$

Similarly we can check that

$$u_n(x)=J_x^n(\hat{V}_n),$$

and the result has been obtained. □

We can then state

Theorem 4.4. *Under the assumptions* (4.3.1), (4.3.2), (4.3.6) *the minimum solution* $\underline{u}(x)$ *of the functional equation* (4.3.7) *between* 0 *and* w_0 *is the value function*

$$\underline{u}(x)=\inf_V J_x(V).$$

Moreover there exists an optimal control policy $\hat{V}$.

PROOF. For any decision rule V, we have according to Lemma 4.4,

$$u_n(x)\le J_x^n(V)\le J_x(V).$$

Therefore

$$\underline{u}(x)\le J_x(V),$$

and since V is arbitrary

$$\underline{u}(x)\le\inf_V J_x(V).$$

On the other hand, since $\underline{u}(x)$ is l.s.c. positive, there exists a Borel map $\hat{v}(x)$ such that

$$\underline{u}(x)=l(x,\hat{v}(x))+\alpha\Phi^{\hat{v}}\underline{u}(x).$$

One then defines a control policy$\hat{V}$ as follows

$$\hat{V}=(\hat{v}_1,\cdots,\hat{v}_n,\cdots),$$

where

$$\hat{v}_n(x_1,\cdots,x_n)=\hat{v}_n(x_n).$$

We obtain easily

$$\underline{u}(x)=\sum_{j=1}^n E^{\hat{V},x}\alpha^{j-1}l(y_j,\hat{v}_j)+\alpha^n E^{\hat{V},x}\underline{u}(y_{n+1}),$$

and since $\underline{u} \geq 0$ we have

$$\underline{u}(x) \geq \sum_{j=1}^{n} E^{\hat{V},x} \alpha^{j-1} l(y_j, \hat{v}_j).$$

Letting $n \to \infty$ yields

$$\underline{u}(x) \geq J_x(\hat{V}),$$

which implies in fact equality. The result has been obtained. □

4.4.2. THE MAXIMUM SOLUTION. We turn now to the interpretation of the maximum solution $\overline{u}(x)$. We begin by interpreting the function $u^n(x)$ obtained by the decreasing sequence

$$\begin{aligned} u^0(x) &= w_0(x), \\ u^{n+1}(x) &= \inf_{v \in U} [l(x,v) + \alpha \Phi^v u^n(x)]. \end{aligned} \tag{4.4.3}$$

We define a set of control policies as follows

$$\begin{aligned} &\mathcal{V}^n = \{V | V = (v_1, \cdots, v_j, \cdots) \text{ where} \\ &v_j = v_j(x_1, \cdots, x_j) \forall j \leq n, \\ &v_j = v_0 \; \forall j \geq n+1\}, \end{aligned} \tag{4.4.4}$$

where the functions v_j are Borel functions with values in U. We then state the

Lemma 4.5. *We have the interpretation*

$$u^n(x) = \inf_{V \in \mathcal{V}^n} J_x(V) \tag{4.4.5}$$

PROOF. By a reasoning similar to that of Lemma 4.4 we can write, for an arbitrary control policy V (not necessarily in $\mathcal{V}^n$)

$$u^n(x) \leq E^{V,x} \sum_{j=1}^{n} \alpha^{j-1} l(y_j, v_j) + E^{V,x} \alpha^n w_0(y_{n+1}).$$

Take now $V \in \mathcal{V}^n$. We can write for $m \geq n$

$$E^{V,x}[\mathbb{I}_\Gamma(y_{m+1}) | \mathcal{Y}^m] = \Phi^{v_n(y_1, \cdots, y_n)} \mathbb{I}_\Gamma(y_m) = \Phi^{v_0} \mathbb{I}_\Gamma(y_m).$$

Considering next w_0, we can write for $j \geq n+1$

$$w_0(y_j) = l(y_j, v_0) + \alpha E^{V,x}[w_0(y_{j+1}) | \mathcal{Y}^j].$$

In particular

$$E^{V,x} w_0(y_{n+1}) = E^{V,x} \sum_{j=n+1}^{\infty} \alpha^{j-n-1} l(y_j, v_0),$$

hence also

$$u^n(x) \leq E^{V,x} \sum_{j=1}^{\infty} \alpha^{j-1} l(y_j, v_j), \forall V \in \mathcal{V}^n.$$

Next we note that the functions $u^n(x)$ are l.s.c. hence we can define a Borel map $v^n(x)$ such that

$$u^n(x) = l(x, v^n(x)) + \alpha \Phi^{v^n(x)} u^n(x).$$

One then defines

$$\hat{V}^n = (\hat{v}_1^n, \cdots, \hat{v}_j^n, \cdots),$$

with

$$\begin{array}{l}\hat{v}_j^n(x_1,\cdots,x_j)=v^j(x_j),\forall j\leq n\\ \hat{v}_j^n(x_1,\cdots,x_j)=v_0,\forall j\geq n+1\end{array}.$$

We obtain

$$u^n(x)=E^{\hat{V}^n,x}\sum_{j=1}^{n}\alpha^{j-1}l(y_j,\hat{v}_j^n)+\alpha^nE^{\hat{V}^n,x}w_0(y_{n+1}).$$

Therefore

$$u^n(x)=E^{\hat{V}^n,x}\sum_{j=1}^{\infty}\alpha^{j-1}l(y_j,\hat{v}_j^n)$$

$$u^n(x)=J_x(\hat{V}^n).$$

Since $\hat{V}^n\in\mathcal{V}^n$ the proof of the Lemma is completed. □

To finally give the interpretation of the maximum solution we introduce the set

$$\begin{array}{l}\mathcal{V}=\{V|V=(v_1,\cdots,v_j,\cdots)\text{ where}\\ v_j=v_j(x_1,\cdots,x_j)\\ \alpha^{j-1}E^{V,x}\overline{u}(y_j)\to 0\text{ as }j\to+\infty\}.\end{array}\tag{4.4.6}$$

We can then state the

Theorem 4.5. *Under the assumptions* (4.3.1), (4.3.2), (4.3.6) *the maximum solution* $\bar{u}(x)$ *of the functional equation* (4.3.7) *between* 0 *and* w_0 *is given by*

$$\overline{u}(x)=\inf_{V\in\mathcal{V}}J_x(V)\tag{4.4.7}$$

Proof. Consider the control policy $\hat{V}^n$. We are going to check that it belongs to $\mathcal{V}$, for any n. We need to check that

$$\alpha^{k-1}E^{\hat{V}^n,x}\overline{u}(y_k)\to 0\text{ as }k\to+\infty.$$

We may assume $k\geq n+1$. Since $\overline{u}(x)\leq w_0(x)$ it is sufficient to check that

$$\alpha^{k-1}E^{\hat{V}^n,x}w_0(y_k)\to 0\text{ as }k\to+\infty.$$

However

$$\alpha^{k-1}E^{\hat{V}^n,x}w_0(y_k)=E^{\hat{V}^n,x}\sum_{j=k}^{\infty}\alpha^{j-1}l(y_j,v_0),$$

for $k\geq n+1$.

But the series

$$E^{\hat{V}^n,x}\sum_{j=n+1}^{\infty}\alpha^{j-1}l(y_j,v_0)\leq u^n(x)<\infty,$$

and thus is a convergent series. Therefore indeed

$$E^{\hat{V}^n,x}\sum_{j=k}^{\infty}\alpha^{j-1}l(y_j,v_0)\to 0\text{ as }k\to\infty.,$$

and the result follows. Next we know that

$$u^n(x)=J_x(\hat{V}^n).$$

Therefore

$$u^n(x) \geq \inf_{V \in \mathcal{V}} J_x(V),$$

and also

$$\bar{u}(x) \geq \inf_{V \in \mathcal{V}} J_x(V).$$

On the other hand for any control, not necessarily in $\mathcal{V}$ we can write

$$\begin{aligned}\bar{u}(x) &\leq E^{V,x} \sum_{j=1}^{n} \alpha^{j-1} l(y_j, v_j) + \alpha^n E^{V,x} \bar{u}(y_{n+1}) \\ &\leq J_x(V) + \alpha^n E^{V,x} \bar{u}(y_{n+1}).\end{aligned}$$

Now if $V \in \mathcal{V}$ the 2nd term goes to 0 as $n \to \infty$. Therefore we can assert

$$\bar{u}(x) \leq J_x(V), \forall V \in \mathcal{V},$$

and this completes the proof of the Theorem. □

Remark. We do not claim that the infimum is attained in the right hand side of (4.4.7). Moreover the maximum solution is u.s.c. and not l.s.c.

4.5. UNIQUENESS

We consider the problem of uniqueness of solutions of the functional equation (4.3.7), in the interval $(0, w_0)$. A necessary and sufficient condition for uniqueness is that the minimum and the maximum solutions coincide. To prove that the minimum and maximum solutions coincide, it is sufficient to prove that the optimal control $\hat{V} \in \mathcal{V}$. A different approach is to use contraction properties, when they are available.

Theorem 4.6. *Assume $l(x, v)$ bounded. Then there exists one and only solution of* (4.3.7)

Proof. We can use a contraction property. Let us recall that B denotes the space of bounded functions on $X = R^d$. We notice that (4.3.7) can be written as a fixed point for the map

$$T : B \to B, \ z = Tw,$$

with

$$Tw(x) = \inf_{v \in U} [l(x, v) + \alpha \Phi^v w(x)],$$

and since l is bounded we can define T as a map from B into B. With $\alpha < 1$ it is obvious that T is a contraction. Hence there exists one and only one fixed point. □

Remark. Since w_0 is bounded, the uniqueness follows also immediately from the fact that condition (4.4.6) is always satisfied.

This result can be extended in several ways.

Theorem 4.7. *Assume that*

$$0 \leq l(x, v) \leq h|x| + c|v| + l_0, \tag{4.5.1}$$

$$\Phi^v(|x|)(x) \leq \theta|x| + \lambda|v| + \mu, \tag{4.5.2}$$

and U is compact. If $\alpha\theta < 1$ the solution of (4.3.7) *in the space B_1 of functions with linear growth exists and is unique.*

PROOF. In equation (4.5.2) we have denoted by $|x|$ the function $x \to |x|$. We provide B_1 with an equivalent norm

$$||u||_1 = \sup \frac{|u(x)|}{1+\sigma|x|},$$

where σ can be chosen arbitrarily and will be chosen small. For $v \in U$ we have

$$l(x,v) \leq h|x| + l_1$$

$$\Phi^v(|x|)(x) \leq \theta|x| + m$$

Consider again the map T defined in the proof of Theorem, this time from B_1 into B_1. Note that for $w \in B_1$ and $v \in U$ we have

$$\begin{aligned} \alpha|\Phi^v w(x)| &\leq \alpha||w||(1+\sigma\Phi^v(|x|)(x)) \\ &\leq \alpha||w||(1+\sigma\theta|x|+\sigma m). \end{aligned}$$

Then

$$||Tw_1 - Tw_2|| \leq \alpha||w|| \sup_x \frac{1+\sigma\theta|x|+\sigma m}{1+\sigma|x|}.$$

Thanks to the assumptions we have

$$\alpha \sup_x \frac{1+\sigma\theta|x|+\sigma m}{1+\sigma|x|} \leq \alpha(1+\sigma m),$$

and by choosing σ sufficiently small the number on the right hand side is strictly smaller than 1. Hence T is a contraction in B_1. The existence and uniqueness follows. □

Remark. Again it is possible to see the preceding result as a consequence of the fact that the maximum and the minimum solutions coincide. Indeed, take to fix ideas $v_0 = 0$. We note first that from the growth assumptions we have

$$w_0(x) \leq \frac{h|x|}{1-\alpha\theta} + \frac{l_0}{1-\alpha} + \frac{\alpha\mu}{1-\alpha\theta}. \tag{4.5.3}$$

Take any control policy V. From the definition of $\mathcal{V}$, see (4.4.6), using the fact that $\bar{u} \leq w_0$, it is sufficient to prove that

$$\alpha^{j-1}E^{V,x}|y_j| \to 0 \text{ as } j \to +\infty.$$

However

$$\begin{aligned} E^{V,x}[|y_{j+1}|\,|\mathcal{Y}_j] &= \Phi^{v_j}|x|(y_j) \\ &\leq \theta|y_j| + m. \end{aligned}$$

hence

$$E^{V,x}|y_{j+1}| \leq \theta E^{V,x}|y_j| + m,$$

from which it follows that

$$\begin{aligned} E^{V,x}|y_j| &\leq \theta^{j-1}|x| + m\sum_{k=0}^{j-2}\theta^k \\ &\leq \theta^{j-1}|x| + m(j-1)\theta^{j-2}, \end{aligned}$$

since $\theta \geq 1$ without loss of generality. It follows that $\alpha^{j-1}E^{V,x}|y_j| \to 0$ as $j \to \infty$. Therefore the maximum and minimum solutions coincide and there is uniqueness.

In the case when U is not compact it is possible to obtain uniqueness in the following case

$$\begin{aligned} &h|x| + c|v| \le l(x,v) \\ &\Phi^v(\Phi^0)^{n-1}|x|(x) \ge (\Phi^0)^n|x|(x), \forall n, \forall v \in U \end{aligned} \tag{4.5.4}$$

Theorem 4.8. *We make the assumptions of Theorem 4.7, except that U compact is replaced by* (4.5.4)*. Then we have existence and uniqueness of the solution of the functional equation* (4.3.7) *in* B_1

PROOF. The proof consists in finding an a priori bound for v in the right hand side of equation (4.3.7). We shall prove that it is sufficient to take $v \in U$ such that

$$|v| \le \frac{l_0}{c(1-\alpha)},$$

and then we can assume that U is compact. Indeed, from (4.3.7) and the assumption on $l(x, v)$ we can check that a solution $u(x)$ must satisfy

$$u(x) \ge h \sum_{j=1}^{\infty} (\alpha\Phi^0)^{j-1}|x|(x).$$

Therefore

$$l(x,v) + \alpha\Phi^v u(x) \ge c|v| + h \sum_{j=1}^{\infty} (\alpha\Phi^0)^{j-1}|x|(x),$$

and thus it is sufficient to look for v such that

$$c|v| + h \sum_{j=1}^{\infty} (\alpha\Phi^0)^{j-1}|x|(x) \le w_0(x).$$

But (recall that $v_0 = 0$)

$$w_0(x) \le \frac{l_0}{1-\alpha} + h \sum_{j=1}^{\infty} (\alpha\Phi^0)^{j-1}|x|(x),$$

and thus it is sufficient to pick $v \le \frac{l_0}{c(1-\alpha)}$. □

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CHAPTER 5

INVENTORY CONTROL WITHOUT SET UP COST

In this Chapter, we apply the general results of Chapter 4 to the case of Inventory Control. We will consider situations when shortage is not allowed (No backlog) and when shortage is allowed (Backlog). In the case when shortage is not allowed, the demand which cannot be met is lost, and a penalty is incurred. The situation of set up costs will be considered in Chapter 9.

5.1. NO SHORTAGE ALLOWED.

5.1.1. STATING OF THE PROBLEM. We are in the situation of section 4.2.3 of Chapter 4. If y_n denotes the inventory at time n, we have the evolution

$$y_{n+1} = (y_n + v_n - D_n)^+ \quad y_1 = x, \tag{5.1.1}$$

where v_n is the control at time n and D_n is the demand, which is a sequence of independent random variables, with probability density $f(x)$. We denote by $F(x)$ the CDF, cumulated distribution function associated to $f(x)$. The control policy, called V, is defined by a sequence of measurable functions $v_n(D_1, \cdots D_{n-1})$. For $n = 1$, v_1 is simply a deterministic number. Introducing the $\sigma-$algebra $\mathcal{F}^n = \sigma(D_1, \cdots D_{n-1})$,$v_n$ is $\mathcal{F}^n$ measurable. The operator Φ^v is given by

$$\Phi^v \varphi(x) = E\varphi((x + v - D)^+). \tag{5.1.2}$$

The transition probability is given by

$$\pi(x, v; d\eta) = \overline{F}(x + v)\delta(\eta) + f(x + v - \eta)\mathbb{1}_{\eta<x+v} d\eta, \tag{5.1.3}$$

hence

$$\Phi^v \phi(x) = \overline{F}(x + v)\phi(0) + \int_0^{x+v} f(x + v - \eta)\phi(\eta)d\eta. \tag{5.1.4}$$

The set $U = [0, \infty)$ is not compact. The image of a test function $\phi(x, v)$ by the operator Φ^v is given by

$$\Phi^v \phi_v(x) = \phi(0, v)\overline{F}(x + v) + \int_0^{x+v} \phi(x + v - \eta, v)f(\eta)d\eta, \tag{5.1.5}$$

and the assumption (4.3.1) is satisfied.

We define the function $l(x, v)$ as follows

$$l(x, v) = cv + hx + p\int_{x+v}^{\infty} (\xi - x - v)f(\xi)d\xi. \tag{5.1.6}$$

It incorporates an ordering cost (purely a variable cost), a holding cost and a penalty for lost sales. We note the inequalities

$$cv + hx \le l(x, v) \le cv + hx + p\bar{D}, \tag{5.1.7}$$

where $\bar{D} = \int_0^\infty \xi f(\xi) d\xi$. In fact

$$l(x,v) = cv + hx + pE(D - x - v)^+, \tag{5.1.8}$$

and the last term is the penalty cost incurred when the demand is not met. Note that the assumption (4.5.4) is satisfied.

Moreover

$$\Phi^v l_v(x) \le cv + hx + p\bar{D} + hv.$$

More generally

$$(\Phi^v)^n l_v(x) \le cv + hx + p\bar{D} + nhv.$$

Also, denoting by x, the identity function $x \to x$, we have

$$\begin{aligned} \Phi^v x(x) &= E(x + v - D)^+ \\ &\le x + v, \end{aligned} \tag{5.1.9}$$

hence the assumption (4.5.2) is satisfied, with $\theta = 1$ and $\lambda = 1, \mu = 0$.

5.1.2. BELLMAN EQUATION. We can apply the results of Theorem 4.8. The functional equation (4.3.7) reads

$$u(x) = \inf_{v \ge 0} \left\{ l(x,v) + \alpha \left(u(0)\bar{F}(x+v) + \int_0^{x+v} u(x+v-\eta) f(\eta) d\eta \right) \right\}. \tag{5.1.10}$$

It is convenient to look at (5.1.10) as a system for a pair $u(.), \lambda$ where $u(.)$ is a locally integrable function with linear growth and λ is a positive real number. The system is written as

$$\begin{aligned} u(x) &= \inf_{v \ge 0} \left\{ l(x,v) + \alpha \left(\lambda \bar{F}(x+v) + \int_0^{x+v} u(x+v-\eta) f(\eta) d\eta \right) \right\} \\ \lambda &= \inf_{v \ge 0} \left\{ l(0,v) + \alpha \left(\lambda \bar{F}(v) + \int_0^{v} u(v-\eta) f(\eta) d\eta \right) \right\} \end{aligned} \tag{5.1.11}$$

Naturally, the solution $u(x)$ is continuous and $u(0) = \lambda$. Moreover, the argument v of the infimum in the right hand side of (5.1.11) can be taken bounded by $\dfrac{p\bar{D}}{c(1-\alpha)}$. Recall the definition of B_1, space of functions with linear growth, which in the present situation, where the argument lies in R^+ reduces to

$$u \in B_1 \Longleftrightarrow \frac{|u(x)|}{1+x} \le C.$$

We can assert, as a consequence of Theorem 4.8 that

Theorem 5.1. *Assume* (5.1.2) *and* (5.1.6)*. Then the system* (5.1.11) *has a unique solution in the space* $B_1 \times R$*. The function* $u(x)$ *is continuous and* $\lambda = u(0)$*. Moreover* $u(x)$ *is the value function of the control problem*

$$u(x) = \inf_V J_x(V),$$

where

$$J_x(V) = E\left[\sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n) \right]. \tag{5.1.12}$$

Furthermore there exists an optimal control, obtained from a feedback $\hat{v}(x)$*, which is a Borel function attaining the infimum in the right hand side of* (5.1.11)*. One has* $\hat{v}(x) \le \dfrac{p\bar{D}}{c(1-\alpha)}$.

Remark. We have not used the construction of $P^{V,x}$ developed for general Markov chains in section 4.2.1, since we have the formulation described in section 4.2.3.

5.1.3. BASE STOCK POLICY. In this section we will check that $\hat{v}(x)$ is actually quite simple, and is characterized by what is called a *Base Stock Policy.* A Base Stock Policy is characterized by a number S and the formula

$$\hat{v}(x) = \begin{cases} S - x, & \text{if } x \leq S \\ 0, & \text{if } x \geq S \end{cases} \tag{5.1.13}$$

Theorem 5.2. *Under the assumptions* (5.1.2) *and* (5.1.6)*, and* $p > c$ *the optimal feedback is a base stock policy. The function* u *is convex and* C^1*. The base stock is given by the formula*

$$\bar{F}(S) = \frac{c(1-\alpha) + \alpha h}{p - \alpha c + \alpha h}. \tag{5.1.14}$$

One has

$$u(S) = (h-c)S + \frac{cS + pE(S-D)^-}{1-\alpha} + (h-c)\frac{\alpha}{1-\alpha}E(S-D)^+. \tag{5.1.15}$$

Setting

$$g(x) = \big(c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x)\big)^+, \tag{5.1.16}$$

then $u'(x) = h - c + z(x)$ *where*

$$z(x) = \sum_{n=1}^{\infty} \alpha^{n-1} g \star f^{\star(n-1)}, \tag{5.1.17}$$

and $\varphi \star \psi$ *denotes the convolution product*

$$\varphi \star \psi(x) = \int_0^x \varphi(x-\xi)\psi(\xi)\, d\xi,$$

and $f^{\star(0)}(x) = \delta(x),\ f^{\star(1)}(x) = f(x)$.

Proof. Note that S is uniquely defined, thanks to the assumption $p > c$. Moreover $g(x) = 0$, for $x \leq S$. Consequently $z = 0$, for $x \leq S$. The series $z(x)$ converges since f is bounded.

Recalling that

$$l(x, v) = cv + hx + pE(x + v - D)^-,$$

we can write also

$$l(x, v) = (h-c)x + l_0(x+v),$$

with

$$l_0(x) = cx + pE(x-D)^-.$$

Therefore equation (5.1.11) can be also written as

$$u(x) = (h-c)x + \inf_{v \geq 0}\{l_0(x+v) + \alpha E u((x+v-D)^+)\}. \tag{5.1.18}$$

The function $l_0(x)$ is convex continuous on R^+ and tends to $+\infty$ as $x \to +\infty$.

There exists a unique S_0 such that

$$l_0(S_0) = \inf_{x \geq 0} l_0(x).$$

Consider the increasing sequence. We begin with $u_1(x)$. Clearly we can write

$$u_1(x) = \begin{cases} (h-c)x + l_0(S_0), & \forall x \le S_0 \\ (h-c)x + l_0(x), & \forall x \ge S_0 \end{cases}$$

Its derivative is

$$(u_1)'(x) = \begin{cases} h-c, & \forall x < S_0 \\ h-c+(l_0)'(x), & \forall x > S_0 \end{cases}$$

This function is continuous and increasing, from which it follows that $u_1(x)$ is convex continuous and tends to $+\infty$ as $x \to +\infty$. More generally, assume that X

$$\varsigma_n(x) = l_0(x) + \alpha E u_n((x-D)^+),$$

is convex continuous and tends to $+\infty$ as $x \to +\infty$. Therefore there exists a unique S_n defined as the smallest minimum value of the function $\varsigma_n(x)$, for $x \ge 0$, if the minimum is not unique. Hence we can write

$$u_{n+1}(x) = \begin{cases} (h-c)x + \varsigma_n(S_n), & \forall x \le S_n \\ (h-c)x + \varsigma_n(x), & \forall x \ge S_n \end{cases}$$

It follows that

$$u'_{n+1}(x) = h - c + \varsigma'_n(x) \mathbb{1}_{x > S_n},$$

which is an increasing function of x. Therefore $u_{n+1}(x)$ is also a convex continuous function which tends to $+\infty$ as $x \to +\infty$. Consider then $\varsigma_{n+1}(x)$. We have

$$\begin{aligned} \varsigma'_{n+1}(x) &= l'_0(x) + \alpha E u'_{n+1}(x-D) \mathbb{1}_{x>D} \\ &= l'_0(x) + \alpha(h-c)F(x) + \alpha E \varsigma'_n(x-D) \mathbb{1}_{x-D>S_n} \\ &= c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x) + \alpha E \varsigma'_n(x-D) \mathbb{1}_{x-D>S_n}, \end{aligned}$$

and this function is increasing in x. Therefore $\varsigma_{n+1}(x)$ is convex continuous and tends to $+\infty$ as $x \to +\infty$.

Since all the u_n, ς_n functions are convex, the limits u, ς are also convex, with

$$\varsigma(x) = l_0(x) + \alpha E u((x-D)^+).$$

Being convex and finite for any x, they are also continuous for $x > 0$. They also tend to $+\infty$ as as $x \to +\infty$. So the optimal feedback is defined by a base stock S which is the smallest minimum of $\varsigma(x)$.

Next, from (5.1.18) we can write

$$u(x) = \begin{cases} (h-c)x + l_0(S) + \alpha E u((S-D)^+), & \forall x \le S \\ (h-c)x + l_0(x) + \alpha E u((x-D)^+), & \forall x \ge S \end{cases}$$

Since $(S-D)^+ \le S$ we can easily calculate

$$Eu((S-D)^+) = \frac{(h-c)E(S-D)^+ + l_0(S)}{1-\alpha},$$

therefore formula (5.1.15) follows. Next

$$\varsigma'(x) = l'_0(x) + \alpha E[u'((x-D)^+) \mathbb{1}_{x>D}].$$

We express the fact that $\varsigma'(S) = 0$. But $u'((S-D)^+) = h - c$. Then $\varsigma'(S)$ can be easily expressed. The formula (5.1.14) follows easily. It follows also that $u'(x)$ is continuous in S, and thus $u(x)$ is C^1. We next define

$$\begin{aligned} z(x) &= u'(x) - (h-c) \\ &= \varsigma'(x)\mathbb{1}_{x>S}. \end{aligned}$$

Differentiating the expression above of $u(x)$ for $x \geq S$ we obtain

$$z(x) = l_0'(x) + \alpha(h-c)F(x) + \alpha E[z(x-D)\mathbb{1}_{x>D}].$$

Noting that $z(x) = 0, \forall x \leq S$, we can write this equation as

$$z(x) = g(x) + \alpha E z(x-D),$$

and the solution of this equation is given by the series (5.1.17). This completes the proof. □

We have introduced the condition $p > c$. Let us consider now the situation in which $c \geq p$.

Theorem 5.3. *Under the assumptions* (5.1.2) *and* (5.1.6) *and* $p \leq c$ *then* $u(x) = w_0(x)$. *Hence* $\hat{v}(x) = 0$.

PROOF. Consider the approximation of u by the increasing sequence. We have

$$u_1(x) = (h-c)x + \inf_{v \geq 0}\{l_0(x+v)\},$$

but, from the assumption $l_0(x)$ is monotone increasing. Therefore

$$u_1(x) = (h-c)x + l_0(x).$$

Then

$$u_2(x) = (h-c)x + \inf_{v \geq 0}\{l_0(x+v) + \alpha E u_1((x+v-D)^+)\}.$$

But

$$l_0(x) + \alpha E u_1((x-D)^+) = l_0(x) + \alpha(h-c)E(x-D)^+ + \alpha E l_0((x-D)^+).$$

Since l_0 is monotone increasing, the last term on the right hand side is monotone increasing. If we take the derivative of the first two terms we get

$$l_0'(x) + \alpha(h-c)F(x) = c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x) \geq 0.$$

Therefore $l_0(x) + \alpha E u_1((x-D)^+)$ is monotone increasing and again the infimum is attained at $v = 0$, and we have

$$u_2(x) = (h-c)x + l_0(x) + \alpha E u_1((x-D)^+).$$

By induction, we check that $l_0(x) + \alpha E u_n((x-D)^+)$ is monotone increasing and

$$u_{n+1}(x) = (h-c)x + l_0(x) + \alpha E u_n((x-D)^+).$$

Therefore

$$u(x) = (h-c)x + l_0(x) + \alpha E u((x-D)^+),$$

hence $u(x) = w_0(x)$ and $\hat{v}(x) = 0$. The proof has been completed. □

Remark 5.1. A Base Stock policy satisfies $\hat{v}(x) \leq S$. We have seen in Theorem 5.1 that $\hat{v}(x) \leq \dfrac{p\bar{D}}{c(1-\alpha)}$. Let us check that $S \leq \dfrac{p\bar{D}}{c(1-\alpha)}$. This follows from formula (5.1.14). Indeed we have

$$\bar{D} \geq S\bar{F}(S) = S\,\frac{c(1-\alpha)+\alpha h}{p-\alpha c+\alpha h} \geq S\,\frac{c(1-\alpha)}{p},$$

and the result follows.

Suppose $x \leq S$. If we apply a Base stock policy, then the stock will never be larger than S. Therefore we order at each period and we have

$$\begin{aligned} y_n + v_n &= S \qquad n \geq 1 \\ y_n &= (S - D_{n-1})^+ \quad n \geq 2 \end{aligned}$$

So the cost of the first period is $(h-c)x + l_0(S)$ and the cost of all the following periods is the same and is equal to $(h-c)E(S-D)^+ + l_0(S)$. Therefore the global cost is

$$u(x) = (h-c)x + l_0(S) + \frac{\alpha}{1-\alpha}[(h-c)E(S-D)^+ + l_0(S)],$$

and we recover what has been obtained by Dynamic Programming, and by differentiating in S, we recover the value in (5.1.14). This fast way to obtain the right base stock is not an alternative to Dynamic Programming, since it does not prove that a Base stock policy is optimal. It is just an optimization among base stock policies over the value of S. We can express v_n, for $n \geq 2$, by

$$v_n = \min(S, D_{n-1}).$$

This approach could be misleading, if we take $x = S$. One could be tempted to minimize to minimize in S the quantity

$$(h-c)S + l_0(S) + \frac{\alpha}{1-\alpha}[(h-c)E(S-D)^+ + l_0(S)],$$

which would lead to a wrong value.

5.2. BACKLOG ALLOWED

5.2.1. STATING OF THE PROBLEM. We consider the following evolution of the inventory y_n

$$y_{n+1} = y_n + v_n - D_n \quad y_1 = x, \tag{5.2.1}$$

where the demand D_n is a sequence of independent random variables with probability density function $f(x)$ and CDF $F(x)$. The operator Φ^v is defined by

$$\Phi^v\varphi(x) = E\phi(x+v-D) = \int_{-\infty}^{x+v} f(x+v-\eta)\phi(\eta)d\eta, \tag{5.2.2}$$

and the transition probability is given by

$$\pi(x,v;d\eta) = f(x+v-\eta)\mathbb{1}_{\eta<x+v}d\eta.$$

The control policy v_n is again a sequence of random variables measurable with respect to the filtration $\mathcal{F}^n = \sigma(D_1,\cdots D_{n-1})$. We introduce the cost per period

$$l(x,v) = cv + hx^+ + px^-, \tag{5.2.3}$$

where c is the variable cost per unit of order, h is the holding per unit cost and p is the shortage per unit cost. We next define the control pay-off

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1} El(y_n, v_n). \tag{5.2.4}$$

The value function $u(x) = \inf\ J_x(V)$ is solution of Bellman equation

$$u(x) = \inf_{v \geq 0}[l(x,v) + \alpha Eu(x+v-D)]. \tag{5.2.5}$$

5.2.2. BELLMAN EQUATION. Consider the ceiling function

$$w_0(x) = \sum_{n=1}^{\infty} \alpha^{n-1} (\Phi^0)^{n-1}(hx^+ + px^-)(x). \tag{5.2.6}$$

It is easy to check that this series is well defined and

$$w_0(x) \leq \max(h,p) \left[\frac{|x|}{1-\alpha} + \frac{\bar{D}}{(1-\alpha)^2} \right]. \tag{5.2.7}$$

Assumptions (4.3.1), (4.3.2), (4.3.6) are satisfied. So results of Theorems 4.3, 4.4, 4.5 are applicable. We can then state

Theorem 5.4. *Under the assumptions* (5.2.2), (5.2.3) *the set of solutions of the functional equation* (5.2.5), *in the interval* $0 \leq u \leq w_0$ *is not empty and has a minimum and a maximum solution* $\underline{u}$ *and* $\overline{u}$. *The minimum solution is l.s.c. It coincides with the value function. There exists an optimal control policy, obtained through a feedback* $\hat{v}(x)$.

We turn now to the question of uniqueness. We can check that the properties (4.5.1), (4.5.2) are satisfied. However, only the first part of property (4.5.4) is satisfied. Therefore we cannot directly apply Theorem 4.8. Hence the situation is not similar to that of Theorem 5.1. The key issue is to obtain estimates on the feedback $\hat{v}(x)$. The feedback will not be bounded, in the case of high negative inventories. We can state the following important result

Proposition 5.1. *The optimal feedback satisfies*

$$\hat{v}(x) \leq x^- + \frac{\alpha \bar{D}}{c(1-\alpha)^2}[h + (p+c)(1-\alpha)]. \tag{5.2.8}$$

PROOF. We begin by getting an estimate from below of the value function. We shall check that

$$u(x) \geq \frac{hx^+}{1-\alpha} + px^- - \frac{h\alpha\bar{D}}{(1-\alpha)^2}. \tag{5.2.9}$$

This estimate is proved by induction. We begin with

$$u(x) \geq hx^+ + px^-,$$

then, using $u(x) \geq hx^+$ in the right hand side of equation (5.2.5), we get

$$\begin{aligned} u(x) &\geq hx^+ + px^- + \alpha hE(x-D)^+ \\ &\geq hx^+ + px^- + \alpha h(x^+ - \bar{D}) \\ &= hx^+(1+\alpha) + px^- - \alpha h\bar{D}. \end{aligned}$$

Iterating again we get

$$u(x) \geq hx^+(1+\alpha+\alpha^2) + px^- - (\alpha + 2\alpha^2)h\bar{D},$$

and in general

$$u(x) \geq hx^+(1+\alpha+\cdots+\alpha^n)+px^- - (\alpha+2\alpha^2+\cdots+n\alpha^n)h\bar{D},$$

and letting $n \to \infty$ one obtains (5.2.9).

The next step is to majorize $u(x)$ by a convenient function. We do not use $w_0(x)$ but the function $w(x)$ solution of

$$w(x) = (c+p)x^- + hx^+ + \alpha E w(x^+ - D). \tag{5.2.10}$$

We claim that $u(x) \leq w(x)$. Indeed (5.2.10) corresponds to the pay off when one uses the feedback x^-. We claim that the following estimate holds

$$w(x) \leq (c+p)x^- + \frac{hx^+}{1-\alpha} + L, \tag{5.2.11}$$

for a convenient constant L. This is checked by induction, but to identify the value of the constant L, we postulate (5.2.10), insert it in the right hand side of (5.2.10). We obtain

$$w(x) \leq (c+p)x^- + \frac{hx^+}{1-\alpha} + \alpha(p+c)\bar{D} + \alpha L,$$

which will be lower than the right hand side of (5.2.11), if

$$L \geq \frac{\alpha(p+c)\bar{D}}{1-\alpha},$$

so we can take the lower bound for L.

Now, going back to the functional equation (5.2.5), we minorize the quantity within brackets on the right hand side, by using

$$u(x) \geq \frac{hx^+}{1-\alpha} - \frac{h\alpha\bar{D}}{(1-\alpha)^2}.$$

We see that it is minorized by

$$cv + \frac{hx^+}{1-\alpha} + px^- - \frac{h\alpha\bar{D}}{(1-\alpha)^2}.$$

Therefore, looking for the infimum, we can restrict the controls v to the set

$$cv + \frac{hx^+}{1-\alpha} + px^- - \frac{h\alpha\bar{D}}{(1-\alpha)^2} \leq w(x).$$

Using then the estimate on $w(x)$, we deduce that we can restrict the set of v to be bounded by the right hand side of (5.2.8). The optimal feedback must satisfy this bound, hence the result (5.2.8).

We can then prove the uniqueness of the solution of (5.2.5) in the space B_1, where

$$u \in B_1 \Longleftrightarrow \frac{|u(x)|}{1+|x|} \leq C.$$

$\square$

Theorem 5.5. *Under the assumptions* (5.2.2), (5.2.3)*, the solution of* (5.2.5) *in the space* B_1 *is unique. Moreover* u *is continuous.*

PROOF. As usual, we are going to check that the minimum and the maximum solution coincide. Define the optimal control associated with the optimal feedback. If we denote the optimal control by $\hat{V} = \{\hat{v}_1, \cdots, \hat{v}_n, \cdots\}$ and the optimal trajectory by $\{\hat{y}_1, \cdots, \hat{y}_n, \cdots\}$ we must prove that $\hat{V}$ belongs to $\mathcal{V}$, which means that

$$\alpha^{j-1} E|\hat{y}_j| \to 0,$$

as $j \to \infty$. However, from the estimate (9.3.8) and setting

$$C = \frac{\alpha \bar{D}}{c(1-\alpha)^2}[h + (p+c)(1-\alpha)],$$

we can state that

$$\hat{y}_{n+1} \le \hat{y}_n^+ + C,$$

hence

$$\hat{y}_{n+1} \le x^+ + nC.$$

On the other hand

$$\hat{y}_{n+1} \ge x - D_1 - \cdots - D_n.$$

Therefore

$$x - n\bar{D} \le E\hat{y}_{n+1} \le x^+ + nC,$$

and the condition is satisfied, hence the result. □

5.2.3. BASE STOCK POLICY. We want to show the following result

Theorem 5.6. *We assume* (5.2.2), (5.2.3) *and*

$$c(1-\alpha) < p\alpha, \tag{5.2.12}$$

then the function u is convex and C^1, except for $x = 0$. The optimal feedback is given by a Base stock policy, with the Base stock S solution of

$$\bar{F}(S) = \frac{c(1-\alpha) + \alpha h}{\alpha(p+h)}. \tag{5.2.13}$$

One has also

$$u(S) = hS + \frac{\alpha}{1-\alpha}[c\bar{D} + hE(S-D)^+ + pE(S-D)^-]. \tag{5.2.14}$$

Define

$$g(x) = \left(c(1-\alpha) + \alpha h - \alpha(h+p)\bar{F}(x)\right)^+, \tag{5.2.15}$$

then the function $z(x) = u'(x) + c - h 1\!\!I_{x>0} + p 1\!\!I_{x<0}$ is given by

$$z(x) = \sum_{n=1}^{\infty} \alpha^{n-1} g \star f^{\star(n-1)}, \tag{5.2.16}$$

with the same notation as in equation (5.1.17)*.*

PROOF. The convexity is proven by considering the monotone increasing sequence

$$u_{n+1}(x) = hx^+ + px^- - cx + \inf_{v \ge 0}[c(x+v) + \alpha E u_n(x+v-D)], \tag{5.2.17}$$

with $u_1(x) = l_0(x) = hx^+ + px^-$. There exists a unique $n_0 \ge 0$ such that

$$\frac{\alpha p}{1-\alpha}(1-\alpha^{n_0+1}) > c > \frac{\alpha p}{1-\alpha}(1-\alpha^{n_0}). \tag{5.2.18}$$

If $n_0 = 0$, it means that $c < \alpha p$. We claim that

$$u_n(x) = \sum_{j=1}^{n} \alpha^{j-1}(\Phi^0)^{j-1} l_0(x) \; , \forall 1 \leq n \leq n_0 + 1.$$

Indeed we have (we use $u_n(x)$ as the definition of the sum in this expression)

$$u_n'(x) = -p\frac{1-\alpha^n}{1-\alpha}, \; \forall x < 0$$

hence

$$c - \alpha p \frac{1-\alpha^n}{1-\alpha} > 0, \; \forall n \leq n_0.$$

Therefore $cx + \alpha E u_n(x-D)$ is increasing for $0 \leq n \leq n_0$, and thus u_n is solution of equation (5.2.17), for $0 \leq n \leq n_0$, which implies the claim.

For $n \geq n_0 + 2$, there exists $S_n > 0$, such that

$$c + \alpha E u_n'(S_n - D) = 0,$$

and

$$u_{n+1}(x) = \begin{cases} hx^+ + px^- - cx + cS_n + \alpha E u_n(S_n - D), & x < S_n \\ hx + \alpha E u_n(x-D), & x \geq S_n \end{cases}$$

This result is proven by induction, noting that $u_n(x)$ is convex and goes to $+\infty$ for $|x| \to +\infty$. Note that u_n is C^1 except for $x = 0$. This proves that the increasing sequence is convex and goes to $+\infty$ for $|x| \to +\infty$. It follows that u is convex and goes to $+\infty$ for $|x| \to +\infty$. Rewriting Bellman equation (5.2.5) as

$$u(x) = hx^+ + px^- - cx + \inf_{v \geq 0}[c(x+v) + \alpha E u(x+v-D)],$$

we see that the optimal feedback is defined by a base stock policy.

The base stock satisfies

$$c + \alpha E u'(S - D) = 0,$$

and

$$Eu(S-D) = \frac{c\bar{D} + hE(S-D)^+ + pE(S-D)^-}{1-\alpha}.$$

Moreover S is the solution of

$$\bar{F}(S) = \frac{c(1-\alpha) + \alpha h}{\alpha(p+h)}.$$

We deduce easily (5.2.14). Next, the solution u can be written as

$$u(x) = \begin{cases} -cx + hx^+ + px^- + cS & \\ \qquad + \frac{\alpha}{1-\alpha}(c\bar{D} + hE(S-D)^+ + pE(S-D)^-), & \forall x \leq S \\ hx^+ + px^- + \alpha E u(x-D), & \forall x \geq S \end{cases}$$

We see that u is C^1 except for $x = 0$. Defining z and g as in the statement of the theorem, we see easily that z is the solution of

$$z(x) = g(x) + \alpha E z(x-D),$$

which is solved by formula (5.2.16). The proof has been completed. □

We can now turn to the case when $c(1-\alpha) \geq \alpha p$. We have

Theorem 5.7. *We assume* (5.2.2), (5.2.3) *and* $c(1-\alpha) \geq \alpha p$. *Then* $u(x) = w_0(x)$, *with*

$$w_0(x) = \sum_{j=1}^{\infty} \alpha^{j-1} (\Phi^0)^{j-1} l_0(x). \tag{5.2.19}$$

Therefore $\hat{v}(x) = 0$.

PROOF. In fact, if we look at the proof of the preceding theorem, the present assumption corresponds to the situation when $n_0 = +\infty$. We then see that the increasing sequence converges to $w_0(x)$. This proves the result. □

Remark 5.2. We have obtained in Proposition 5.1 that $\hat{v}(x) \leq x^- + C$. Moreover from the Base stock policy, we have also $\hat{v}(x) \leq x^- + S$. We must check that $S \leq C$. Using again $\bar{D} \geq S\bar{F}(S)$, we can state that

$$\bar{D} \geq S \frac{c(1-\alpha) + \alpha h}{\alpha(p+h)},$$

hence

$$\frac{C}{S} \geq \frac{[h + (p+c)(1-\alpha)][c(1-\alpha) + \alpha h]}{c(p+h)(1-\alpha)^2},$$

and the constant on the right hand side is bigger than 1.

We have also a remark similar to Remark 5.1

Remark 5.3. Suppose $x \leq S$. If we apply a Base stock policy then we shall put an order at each period, and we have

$$\begin{aligned} y_n + v_n &= S \qquad n \geq 1 \\ y_n &= S - D_{n-1} \qquad n \geq 2 \end{aligned}$$

and thus $v_n = D_{n-1}$. It follows that the cost of a base stock policy is

$$u(x) = hx^+ + px^- + c(S-x) + \frac{\alpha}{1-\alpha}(c\bar{D} + hE(S-D)^+ + pE(S-D)^-),$$

and minimizing this expression in S yields formula (5.2.13).

5.3. DETERMINISTIC CASE

The deterministic case is a particular case of the stochastic case, except for smoothness. We assume that we face a fixed demand D at each period. We briefly describe the results.

5.3.1. NO SHORTAGE ALLOWED.

The inventory evolves as follows

$$y_{n+1} = (y_n + v_n - D)^+ \quad y_1 = x, \tag{5.3.1}$$

and the cost functional is

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n), \tag{5.3.2}$$

with

$$l(x, v) = (h - c)x + l_0(x + v), \tag{5.3.3}$$

and

$$l_0(x) = cx + p(x - D)^-. \tag{5.3.4}$$

The Bellman equation reads

$$u(x) = (h-c)x + \inf_{v\geq 0}\{l_0(x+v) + \alpha u((x+v-D)^+)\}. \tag{5.3.5}$$

The solution $u(x)$ is convex continuous and goes to $+\infty$ as $x \to +\infty$. We have to be careful about differentiability. We are interested in the minimum of $l_0(x)+\alpha u((x-D)^+)$. If S is the minimum, we have for $x \leq S$, $u(x) = (h-c)x + C$, hence for $x \leq S$, we have

$$l_0(x) + \alpha u((x-D)^+) = l_0(x) + \alpha(h-c)(x-D)^+ + \alpha C.$$

Therefore, we must have

$$l_0(x) + \alpha(h-c)(x-D)^+ \geq l_0(S) + \alpha(h-c)(S-D)^+, \quad \forall x < S.$$

Recalling that

$$l_0(x) + \alpha(h-c)(x-D)^+ = cx + p(x-D)^- + \alpha(h-c)(x-D)^+,$$

we see easily that if $p \geq c$, this is achieved in a unique way with $S = D$. If $c > p$ then we have $S = 0$.

Therefore, if $p \geq c$, we have

$$u(x) = (h-c)x + \frac{cD}{1-\alpha}, \forall x \leq D.$$

5.3.2. BACKLOG ALLOWED. In the case of backlog, we have

$$y_{n+1} = y_n + v_n - D \quad y_1 = x, \tag{5.3.6}$$

and the cost functional is

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n), \tag{5.3.7}$$

with

$$l(x,v) = hx^+ + px^- + cv.$$

Bellman equation becomes

$$u(x) = hx^+ + px^- - cx + \inf_{v\geq 0}\{c(x+v) + \alpha u(x+v-D))\}. \tag{5.3.8}$$

We are interested in the minimum of the function $cx + \alpha u(x-D)$. If S is this minimum, we have for $x \leq S$,

$$u(x) = hx^+ + px^- - cx + C,$$

so for $x \leq S$, the function $cx+\alpha[h(x-D)^+ + p(x-D)^- - c(x-D) + C]$ must attain its minimum in S. We obtain that if $\alpha p \geq c(1-\alpha)$ necessarily $S = D$, whereas if $\alpha p < c(1-\alpha)$ necessarily $S = 0$. Therefore

$$u(x) = hx^+ + px^- - cx + \frac{cD}{1-\alpha}, \ \forall x \leq D$$

CHAPTER 6

ERGODIC CONTROL IN DISCRETE TIME

We have in the preceding chapters considered always a control problem with discount $\alpha < 1$. The problem of Ergodic Control corresponds to the case when $\alpha = 1$. This is a limit situation, we cannot just take $\alpha = 1$ in the formulas. The value function becomes infinite. We have studied in Chapter 3 the situation without control. We develop in this Chapter the corresponding theory for controlled Markov chains.

6.1. FINITE NUMBER OF STATES

We study here the simplest case of a Markov chain with a finite number of states.

6.1.1. ASSUMPTIONS-NOTATION. We consider a Controlled Markov chain with a finite number of states. The space of states is denoted by X and the transition probability reduces to a matrix, which is defined by $\varpi^v(i,j) = P(y_{n+1} = j|y_n = i, v)$, where y_n represents the canonical process (the state of the system at time n). The control $v \in U$. We assume

$$\begin{aligned} &U \text{ compact} \\ &\varpi^v(i,j) \text{ is continuous in } v, \forall i,j \\ &\varpi^v(i,j) > 0, \forall i,j,v \end{aligned} \tag{6.1.1}$$

Note that this assumption implies a uniform bound below, since we have a finite number of functions which are strictly continuous on a compact. We write

$$\varpi^v(i,j) \geq \delta > 0. \tag{6.1.2}$$

We denote by Φ^v the operator associated with the matrix ϖ^v. We recall that

$$\Phi^v f(i) = \sum_j \varpi^v(i,j) f(j).$$

This formula extends to a feedback $v(.)$. We will define $\Phi^{v(.)} f$ by

$$\Phi^{v(.)} f(i) = \sum_j \varpi^{v(i)}(i,j) f(j).$$

To the operator $\Phi^{v(.)}$ is associated the transition matrix $\varpi^{v(i)}(i,j)$. We clearly have the property

$$\varpi^{v(i)}(i,j) \geq \delta, \forall i,j \tag{6.1.3}$$

6.1.2. ERGODIC PROPERTIES. Thanks to (6.1.3) the property (3.5.2) in Chapter 3 is satisfied, for any given feedback. Therefore there will exist a unique invariant measure denoted by $m^{v(.)}$ such that

$$\sum_i m_i^{v(.)} \varpi^{v(i)}(i,j) = m_j^{v(.)}, \forall j. \tag{6.1.4}$$

Note that this invariant measure is not related to a specific value of the control, but combines all possible values of the control. There is an averaging effect taking account of all the values $v(i)$. For any $f \in B$ we have the property

$$\left\| (\Phi^{v(.)})^n f - \sum_j m_j^{v(.)} f_j \right\| \leq 2||f||\beta^{n-1}, \forall f \in B, \text{ with } \beta = 1 - \delta \tag{6.1.5}$$

6.1.3. DYNAMIC PROGRAMMING. Consider a function $f(i,v)$ such that

$$l(i,v) \text{ continuous (hence bounded)}, \ l \geq 0. \tag{6.1.6}$$

The Bellman equation of ergodic control is defined by

$$z(i) + \lambda = \inf_{v \in U}[l(i,v) + \Phi^v z(i)]. \tag{6.1.7}$$

We see that there is no α in the right hand side. More precisely, $\alpha = 1$. On the left hand side we have a constant λ, which did not exist in the Bellman equation with discount. In fact we consider the pair $z(.), \lambda$ as the solution of the nonlinear equation (6.1.7). Our main result is the following

Theorem 6.1. *Under the assumptions* (6.1.1) *and* (6.1.6) *there exists one and only one pair* z, λ *which is the solution of* (6.1.7).

PROOF. We consider the Bellman equation with a discount term

$$u_\alpha(i) = \inf_{v \in U}[l(i,v) + \alpha \Phi^v u_\alpha(i)], \tag{6.1.8}$$

and we will let α tend to 1.

This equation admits a unique solution u_α and moreover there exists an optimal feedback $v_\alpha(i)$ such that

$$u_\alpha(i) = l(i, v_\alpha(i)) + \alpha \Phi^{v_\alpha(.)} u_\alpha(i).$$

We can assert that

$$(1-\alpha)||u_\alpha|| \leq ||l|| = \max_{i,v} |l(i,v)|.$$

For any fixed α there exists a unique invariant measure $m^{v_\alpha(.)}$ such that

$$\left\| (\Phi^{v_\alpha(.)})^n \phi - \sum_j m_j^{v_\alpha(.)} \phi_j \right\| \leq 2||\phi||\beta^{n-1}.$$

If we define the function h_α such that

$$h_\alpha(i) = l(i, v_\alpha(i)) - (1-\alpha)\Phi^{v_\alpha(.)} u_\alpha(i),$$

then we have

$$u_\alpha(i) - \Phi^{v_\alpha(.)} u_\alpha(i) = h_\alpha(i),$$

therefore

$$\sum_i m_i^{v_\alpha(.)} h_\alpha(i) = 0.$$

Consider the equation

$$u - \Phi^{v_\alpha(.)}u = h_\alpha,$$

for which by construction u_α is a solution. An other solution is

$$\zeta_\alpha = \sum_{n=1}^{\infty} (\Phi^{v_\alpha(.)})^{n-1} h_\alpha.$$

But two solutions differ only by a constant. So

$$u_\alpha(i) - \zeta_\alpha(i) = K_\alpha, \forall i.$$

It follows that

$$\min_i u_\alpha(i) - \min_i \zeta_\alpha(i) = K_\alpha,$$

hence

$$u_\alpha - \min_i u_\alpha(i) = \zeta_\alpha - \min_i \zeta_\alpha(i).$$

From the expression of h_α one sees that it is bounded as α becomes close to 1. Therefore, from the expression of ζ_α one checks that it is also bounded as α tends to 1. Therefore, we have established that

$$u_\alpha(i) - \min_j u_\alpha(j) \leq C,$$

where the constant does not depend on α nor i. Set

$$z_\alpha = u_\alpha - \min_j u_\alpha(j).$$

It is bounded in α (note that it is positive) and satisfies

$$z_\alpha(i) + \lambda_\alpha = \inf_{v \in U} [l(i,v) + \alpha \Phi^v z_\alpha(i)],$$

with $\lambda_\alpha = (1-\alpha)\min u_\alpha$, also bounded in α.

We can extract subsequences $z_{\alpha_k}, \lambda_{\alpha_k}$ which converge in B for z_{α_k} to a function z and in R for λ_{α_k} to a number λ. It is easy to check that the pair z, λ is a solution of equation (6.1.7).

Let us check uniqueness. Suppose we have two solutions $z_1, \lambda_1, \quad z_2, \lambda_2$. There exist feedbacks $v_1(i), v_2(i)$ such that

$$z_1(i) + \lambda_1 = l(i, v_1(i)) + \Phi^{v_1(.)} z_1(i)$$

$$z_2(i) + \lambda_2 = l(i, v_2(i)) + \Phi^{v_2(.)} z_2(i).$$

Note also that

$$z_2(i) + \lambda_2 \leq l(i, v_1(i)) + \Phi^{v_1(.)} z_2(i).$$

Therefore also

$$z_1(i) - z_2(i) + \lambda_1 - \lambda_2 \geq \Phi^{v_1(.)}(z_1 - z_2)(i).$$

Hence

$$\min_i (z_1(i) - z_2(i)) + \lambda_1 - \lambda_2 \geq \min_i (z_1(i) - z_2(i)).$$

We obtain $\lambda_1 - \lambda_2 \geq 0$ and by symmetry the reverse is also true. Therefore $\lambda_1 = \lambda_2$.

We then have

$$z_1(i) - z_2(i) \geq \Phi^{v_1(.)}(z_1 - z_2)(i),$$

hence also

$$z_1(i) - z_2(i) \geq (\Phi^{v_1(.)})^n (z_1 - z_2)(i).$$

This implies, using the limit of the right hand side

$$z_1(i) - z_2(i) \geq \sum_j m_j^{v_1(.)}(z_1(j) - z_2(j)).$$

If i_0 is the index of minimum of $z_1(i) - z_2(i)$, we have necessarily

$$z_1(i_0) - z_2(i_0) = \sum_j m_j^{v_1(.)}(z_1(j) - z_2(j)).$$

This implies also

$$m_j^{v_1(.)}(z_1(i_0) - z_2(i_0)) = m_j^{v_1(.)}(z_1(j) - z_2(j)), \forall j.$$

Now from the property (6.1.3) we get easily $m_j^{v_1(.)} > \delta, \forall j$ hence

$$z_1(i_0) - z_2(i_0) = z_1(j) - z_2(j), \forall j,$$

which proves the uniqueness of z up to an additive constant. □

6.1.4. EXTENSION. In fact, the assumption (6.1.1) is much too strong. It is sufficient to assume that

$$\begin{array}{l} U \text{ compact} \\ \varpi^v(i,j) \text{ is continuous in } v, \forall i,j \\ \varpi^{v_0}(i,j) > 0, \forall i,j \text{ and some } v_0 \end{array} . \tag{6.1.9}$$

We want to prove the following improvement to Theorem 6.1

Theorem 6.2. *Under the assumptions* (6.1.9) *and* (6.1.6) *there exists one and only one pair* z, λ *which is the solution of* (6.1.7).

PROOF. To the transition matrix $\varpi^{v_0}(i,j)$ corresponds a unique invariant measure $m_i^{v_0}$. These numbers are strictly positive. We can thus consider the inverse

$$q_i^{v_0} = \frac{1}{m_i^{v_0}}.$$

We omit the index v_0 in the following, to simplify the notation. We shall write $\varpi(i,j), m_i, q_i$, and Φ will denote the operator Φ^{v_0}.

Consider the system

$$\begin{array}{l} q_{ij} - \displaystyle\sum_{k \neq j} \varpi(i,k) q_{kj} = 1, \forall j \\ q_{jj} = q_j \end{array} . \tag{6.1.10}$$

We have □

Lemma 6.1. *The system* (6.1.10) *has a unique positive solution.*

PROOF. The system can be rewritten as follows

$$q_{ij} - \sum_k \varpi(i,k) q_{kj} = \theta_{ij},$$

with

$$\theta_{ij} = 1 - \varpi(i,j) q_j,$$

and we have

$$\sum_i m_i \theta_{ij} = 1 - \sum_i m_i \varpi(i,j) q_j = 1 - m_j q_j = 0.$$

From (6.1.5) the result follows. The solution is

$$q = \sum_{n=1}^{\infty} \alpha^{n-1}\Phi^{n-1}\theta.$$

The numbers q_{ij} have a probabilistic interpretation. Indeed q_{ij} represents the average time to reach j (or to come back to i if $j = i$) starting from state i. More precisely we can write

$$q_{ij} = \sum_{n=1}^{\infty} n f^{(n)}(i,j),$$

where $f^{(n)}(i,j)$ is the probability to reach j from i in exactly n steps. We have the recurrence formulas

$$f^{(n)}(i,j) = \sum_{k \neq j} \varpi(i,k) f^{(n-1)}(k,j), n \geq 2$$

$$f^{(1)}(i,j) = \varpi(i,j).$$

The key estimate is given in the following □

Lemma 6.2. *The solution $u_\alpha(i)$ of equation* (6.1.8) *satisfies the estimate*

(6.1.11) $$u_\alpha(i) - u_\alpha(j) \leq ||l|| q_{ij}.$$

Proof. Consider the increasing iteration

$$u_\alpha^{k+1}(i) = \inf_{v \in U} [l(i,v) + \alpha \Phi^v u_\alpha^k(i)],$$

starting with $u_\alpha^0(i) = 0$. We shall check

$$u_\alpha^k(i) - u_\alpha^k(j) \leq ||l|| q_{ij}, \forall k.$$

This inequality is clearly true for $k = 0$. Assume it is true for k, then noting that

$$u_\alpha^{k+1}(i) \leq ||l|| + \alpha \sum_l \varpi(i,l) u_\alpha^k(l),$$

and since all quantities are positive

$$u_\alpha^{k+1}(i) \leq ||l|| + \sum_l \varpi(i,l) u_\alpha^k(l).$$

Using the recurrence we get

$$\begin{aligned} u_\alpha^{k+1}(i) &\leq ||l|| + \sum_{l \neq j} \varpi(i,l)[u_\alpha^k(j) + ||l|| q_{lj}] + \varpi(i,j) u_\alpha^k(j) \\ &= ||l|| + u_\alpha^k(j) + ||l||(q_{ij} - 1) \\ &= u_\alpha^k(j) + ||l|| q_{ij}, \end{aligned}$$

and since the sequence is increasing the property follows.

Applying the Lemma to the index j corresponding to the minimum of $u_\alpha(j)$ and majorizing we obtain also

$$u_\alpha(i) - \min_j u_\alpha(j) \leq ||l|| \max_{ij} q_{ij}.$$

With this estimate the proof can be carried over in the same way as in the proof of Theorem 6.1. □

6.1.5. PROBABILISTIC INTERPRETATION. Our next objective is to interpret the solution of equation (6.1.7). As usual we will define a decision rule

$$V = (v_1, \cdots, v_n, ...),$$

with

$$v_n = v_n(i_1, \cdots, i_n),$$

function of the indices with values in U. To V and i corresponds a probability measure on $\Omega, \mathcal{A}$, the underlying probability space.

We define the cost functions

$$J_i(V) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E^{V,i} l(y_n, v_n),$$

considering the V for which the limit exists. Similarly, define

$$\tilde{J}\,(V) = \lim_{\alpha\to 1} (1-\alpha) \sum_{n=1}^{\infty} \alpha^{n-1} E^{V,i} l(y_n, v_n).$$

We claim the following

Theorem 6.3. *If there exists a solution $z(i), \lambda$ of equation* (6.1.7) *and if there exists a feedback $\hat{v}(i)$ such that*

$$z(i) + \lambda = l(i, \hat{v}(i)) + \Phi^{\hat{v}(.)} z(i),$$

then one has

$$\lambda = \inf_V J_i(V) = \inf_V \tilde{J}_i(V) = J_i(\hat{V}) = \tilde{J}_i(\hat{V}),$$

where $\hat{V}$ is the decision rule associated to the feedback $\hat{v}(.)$.

Proof. Consider any decision rule V. We can write from equation (6.1.7)

$$\lambda + E^{V,i} z(y_n) \leq E^{V,i}[l(y_n, v_n) + z(y_{n+1})],\ \forall n,$$

and for $\hat{V}$ we have

$$\lambda + E^{\hat{V},i} z(y_n) = E^{\hat{V},i}[l(y_n, v_n) + z(y_{n+1})],\ \forall n.$$

Therefore

$$\lambda N + z(x) \leq \sum_{n=1}^{N} E^{V,i} l(y_n, v_n) + E^{V,i} z(y_{N+1}).$$

Hence also

$$\lambda \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E^{V,i} l(y_n, v_n), \forall V,$$

finally

$$\lambda \leq \inf_V \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E^{V,i} l(y_n, v_n) = \inf_V J_i(V).$$

On the other hand we also have

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E^{\hat{V},i} l(y_n, v_n) = J_i(\hat{V}).$$

This relation implies

$$\lambda \geq \inf_V \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} E^{V,i} l(y_n, v_n).$$

Collecting results, we obtain

$$\lambda = \inf_V J_i(V) = J_i(\hat{V}).$$

Similarly from the same initial relations we have also

$$\begin{aligned}\lambda\alpha^{n-1}(1-\alpha) &+ \alpha^{n-1}(1-\alpha)E^{V,i}z(y_n) \\ &\leq \alpha^{n-1}(1-\alpha)E^{V,i}l(y_n,v_n) + \alpha^n(1-\alpha)E^{V,i}z(y_{n+1}) \\ &\quad +\alpha^{n-1}(1-\alpha)^2E^{V,i}z(y_{n+1})\end{aligned}$$

Adding up we obtain

$$\lambda + (1-\alpha)z(i) \leq (1-\alpha)\sum_{n=1}^{\infty} \alpha^{n-1}E^{V,i}l(y_n,v_n) + \sum_{n=1}^{\infty}\alpha^{n-1}(1-\alpha)^2E^{V,i}z(y_{n+1}).$$

Letting $\alpha \to 1$ and using the fact that z is bounded, we deduce

$$\lambda \leq \liminf_{\alpha\to 1}(1-\alpha)\sum_{n=1}^{\infty}\alpha^{n-1}E^{V,i}l(y_n,v_n) = \tilde{J}_i(V).$$

On the other hand one has also

$$\lambda = \lim_{\alpha\to 1}(1-\alpha)\sum_{n=1}^{\infty}\alpha^{n-1}E^{\hat{V},i}l(y_n,v_n) = \tilde{J}_i(\hat{V}),$$

which completes the proof of the desired result. □

6.2. ERGODIC CONTROL OF INVENTORIES WITH NO SHORTAGE

6.2.1. CONVERGENCE RESULTS. We consider the situation of Theorem 5.2. The value function is the unique solution of the functional equation

$$u_\alpha(x) = (h-c)x + \inf_{v\geq 0}[l_0(x+v) + \alpha E u_\alpha((x+v-D)^+)], \tag{6.2.1}$$

where

$$l_0(x) = cx + pE(x-D)^-.$$

We know that the optimal feedback is defined by a Base stock policy S_α with

$$\bar{F}(S_\alpha) = \frac{c+\alpha(h-c)}{p+\alpha(h-c)}, \tag{6.2.2}$$

which has a unique solution since $p > c$. Our objective is to prove the

Theorem 6.4. *We make the assumptions of Theorem 5.2 and $c < p$. Define S uniquely by*

$$\bar{F}(S) = \frac{h}{p+h-c}, \tag{6.2.3}$$

and the number ρ by

$$\rho = p\bar{D} - S(p-c) + (p+h-c)E(S-D)^+. \tag{6.2.4}$$

Define also

$$g(x) = (h - (p + h - c)\bar{F}(x))^+, \tag{6.2.5}$$

then $S_\alpha \uparrow S$ *and*

$$u_\alpha(x) - \frac{\rho}{1-\alpha} \to u(x),$$

where u *is defined by*

$$\begin{aligned} u(S) &= (h-c)(S - E(S-D)^+); \\ u'(x) &= h - c + z(x), \end{aligned} \tag{6.2.6}$$

where z *is defined by*

$$z(x) = \sum_{n=1}^{+\infty} g \star f^{\star(n-1)}, \tag{6.2.7}$$

with the same notation as in equation (5.1.17).

The function $u(x)$ *also satisfies*

$$u(x) = hx + pE(x-D)^- - \rho + Eu((x-D)^+),\ \forall x > S \tag{6.2.8}$$

$$u(x) = hx + c(S-x) + pE(S-D)^- - \rho,\ \forall x \leq S \tag{6.2.9}$$

and also

$$u(x) = (h-c)x - \rho + \inf_{v \geq 0}[l_0(x+v) + Eu((x+v-D)^+)] \tag{6.2.10}$$

$$Eu((S-D)^+) = 0$$

Proof. From the formula (6.2.2) the convergence of S_α to S is easy. Noting that

$$F(S_\alpha) = \frac{p-c}{p+\alpha(h-c)},$$

we see that S_α increases with α. Moreover an easy calculation shows that

$$u_\alpha(S_\alpha) = \frac{1}{1-\alpha}[p\bar{D} + S_\alpha(h(1-\alpha) - p + \alpha c)] + (p + \alpha(h-c))E((S_\alpha - D)^+)],$$

from which it follows, thanks to the choice of ρ that

$$u_\alpha(S_\alpha) - \frac{\rho}{1-\alpha} \to (h-c)(S - E(S-D)^+). \tag{6.2.11}$$

Now we have, from Theorem 5.2

Since

$$u'_\alpha(x) = h - c + z_\alpha(x),$$

where

$$z_\alpha(x) = \sum_{n=1}^{\infty} \alpha^{n-1} g_\alpha \star f^{\star(n-1)}(x),$$

with

$$g_\alpha(x) = \left(c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x)\right)^+.$$

Clearly $g_\alpha(x) \to g(x)$, as $\alpha \to 1$. So $z_\alpha(x) \to z(x)$, defined by (6.2.7), provided this function is well defined, which means that the series converges.

Consider the series

$$H(x) = \sum_{n=2}^{+\infty} f^{\star(n-1)}.$$

We prove by induction that

$$f^{\star(n-1)}(x) \leq ||f||\, F^{n-2}(x), \forall n \geq 2$$

therefore

$$H(x) \leq ||f|| \sum_{n=2}^{\infty} F^{n-2}(x) = \frac{||f||}{\bar{F}(x)}.$$

This bound is well defined for any $x \geq 0$. Note that this requires the implicit assumption that $f(x) > 0, \forall x$. Therefore we can assert that

$$u'_\alpha(x) \to h - c + z(x).$$

Consider $x > S$. We can also assume that $x > S_\alpha$. Therefore

$$\begin{aligned} u_\alpha(x) - u_\alpha(S_\alpha) &= (h-c)(x - S_\alpha) + \int_{S_\alpha}^{z} z_\alpha(\xi)\, d\xi \\ &\to (h-c)(x-S) + \int_{S}^{z} z(\xi)\, d\xi \end{aligned}$$

and from (6.2.8), we deduce

$$u_\alpha(x) - \frac{\rho}{1-\alpha} \to u(x) = (h-c)(S - E(S-D)^+) + (h-c)(x-S) + \int_S^x z(\xi) d\xi. \tag{6.2.12}$$

If $x < S$, we obtain similarly

$$u_\alpha(x) - \frac{\rho}{1-\alpha} \to u(x) = (h-c)(x - E(S-D)^+), \tag{6.2.13}$$

and both formulas coincide for $x = S$, with the first formula (6.2.6). To prove (6.2.9), we write for $x \leq S$,

$$u(S) - u(x) = (h-c)(S-x),$$

and we note that

$$u(S) = hS + pE(S-D)^- - \rho.$$

We then conclude easily. To prove (6.2.8), we note that we can write

$$\begin{aligned} z(x) &= g(x) + Ez(x-D) \\ &= g(x) + E[z(x-D)\mathbb{1}_{x>D}]. \end{aligned}$$

Replacing $z(x)$ by $u'(x) - (h-c)$ we get

$$u'(x) - (h-c) = g(x) + E[u'(x-D)\mathbb{1}_{x>D}] - (h-c)F(x).$$

But we can see that, for $x \geq S$, we have

$$g(x) = c + (h-c)F(x) - p\bar{F}(x),$$

so the previous relation becomes

$$u'(x) = h - pF(x) + E[u'(\bar{x} - D)\mathbb{1}_{x>D}],$$

which can also be written as

$$u'(x) = h + \frac{d}{dx} pE(x-D)^- + \frac{d}{dx} Eu((x-D)^+),$$

Integrating between S and x and using the fact that $Eu((S-D)^+) = 0$, as well as the value of $u(S)$, we conclude easily to obtain (6.2.8).

The proof has been completed. □

6.2.2. PROBABILISTIC INTERPRETATION. Suppose we apply a Base stock policy, with base stock S. Consider $x \leq S$ and the following sequence (where LS_n represents the lost sales of period n

$$\begin{aligned}
&y_1 = x, \;\; v_1 = S - x, \;\; LS_1 = (S - D_1)^-; \\
&y_2 = (S - D_1)^+, \;\; v_2 = S - (S - D_1)^+, \;\; LS_2 = (S - D_2)^+; \\
&\dots \\
&y_{n+1} = (S - D_n)^+, \;\; v_{n+1} = (S - D_n)^+, \;\; LS_{n+1} = (S - D_{n+1})^+.
\end{aligned}$$

Except y_1 all the random variables $y_n, n \geq 2$ have the same probability distribution, which is simply

$$\bar{F}(S)\delta(\eta) + f(S - \eta)\mathbb{I}_{\eta < S} d\eta. \tag{6.2.14}$$

The process y_n is clearly ergodic and the formula (6.2.14) gives the invariant measure.

If we discount with rate α we obtain the cost

$$\begin{aligned}
u_\alpha(x) &= hx + c(S - x) + pE(S - D_1)^- + \\
&\sum_{n=1}^{\infty} \alpha^n E[c(S - (S - D_n)^+) + h(S - D_n)^+ + p(S - D_{n+1})^-],
\end{aligned}$$

therefore

$$u_\alpha(x) = (h - c)(x - E(S - D)^+) + \frac{\rho}{1 - \alpha},$$

and we obtain the value of $u(x)$. The quantity

$$E[c(S - (S - D)^+) + h(S - D)^+ + p(S - D)^-], \tag{6.2.15}$$

represents the cost of one period, except the first one. Minimizing in S gives the optimal base stock. This gives the interpretation of ρ.

Consider now the case $x > S$. We define first

$$y_{n+1} = y_n - D_n, \; y_1 = x,$$

$$LS_n = (y_n - D_n)^-.$$

Let

$$\tau = \inf\{n \geq 1 | y_n \leq S\}.$$

Starting at period τ we are back in the situation where the stock is below S and we proceed as follows

$$\begin{aligned}
&y_\tau = y_\tau, \;\; v_\tau = S - y_\tau, \;\; LS_\tau = (S - D_\tau)^-, \\
&y_{\tau+n} = (S - D_{\tau+n-1})^+, \;\; v_{\tau+n} = S - (S - D_{\tau+n-1})^+, \;\; LS_{\tau+n} = (S - D_{\tau+n})^-.
\end{aligned}$$

The corresponding cost is given by

$$\begin{aligned}
u_\alpha(x) = E &\left[\sum_{n=1}^{\tau-1} \alpha^{n-1}(hy_n + p(y_n - D_{n+1})^-\right] \\
&+ E\alpha^{\tau-1}[(h - c)y_\tau + cS + p(S - D_\tau)^-] \\
&+ \sum_{n=1}^{\infty} E\alpha^{\tau-1+n}[c(S - (S - D_{\tau+n-1})^+) + h(S - D_{\tau+n-1})^+ + p(S - D_{\tau+n})^-],
\end{aligned}$$

hence also by appropriate conditioning

$$u_\alpha(x) - \frac{\rho}{1-\alpha} = E\left[\sum_{n=1}^{\tau-1} \alpha^{n-1}(hy_n + p(y_n - D_{n+1})^- - \rho)\right]$$
$$+E\alpha^{\tau-1}(h-c)(y_\tau - E(S-D)^+)$$

and letting α tend to 1 we obtain

$$u_\alpha(x) - \frac{\rho}{1-\alpha} \to E\left[\sum_{n=1}^{\tau-1} (hy_n + p(y_n - D_{n+1})^- - \rho)\right] +$$
$$+E(h-c)(y_\tau - E(S-D)^+).$$

Replacing $-(h-c)E(S-D)^+$ by $cS + pE(S-D)^- - \rho$, we obtain the probabilistic interpretation of the function $u(x)$ as defined by (6.2.8), (6.2.9).

6.3. ERGODIC CONTROL OF INVENTORIES WITH BACKLOG

6.3.1. CONVERGENCE RESULTS. We consider the situation of Theorem 5.6. The functional equation is given by

$$u_\alpha(x) = -cx + hx^+ + px^- + \inf_{v\geq 0}[c(x+v) + \alpha E u_\alpha(x+v-D)]. \tag{6.3.1}$$

We know that the optimal feedback is defined by a Base stock policy S_α with

$$F(S_\alpha) = \frac{\alpha(p+c)-c}{\alpha(h+p)}, \tag{6.3.2}$$

which has a unique solution provided $\alpha > \frac{c}{p+c}$. This will be assumed since we will let α tend to 1.

Our objective is to prove the

Theorem 6.5. *Define S uniquely by*

$$\bar{F}(S) = \frac{h}{p+h}, \tag{6.3.3}$$

the number ρ by

$$\rho = c\bar{D} + pE(S-D)^- + hE(S-D)^+, \tag{6.3.4}$$

and

$$g(x) = (h - (h+p)\bar{F}(x))^+, \tag{6.3.5}$$

then $S_\alpha \uparrow S$ and

$$u_\alpha(x) - \frac{\rho}{1-\alpha} \to u(x),$$

where u is defined by

$$\begin{array}{l} u(S) = (h-c)\bar{D} - (p+h)E(S-D)^- \\ u'(x) = -c + h1\!\!1_{x>0} - p1\!\!1_{x<0} + z(x) \end{array}, \tag{6.3.6}$$

where

$$z(x) = \sum_{n=1}^{+\infty} g \star f^{\star(n-1)}(x). \tag{6.3.7}$$

The function u is also solution of

$$u(x) = hx - \rho + Eu(x-D), \quad \forall x > S \tag{6.3.8}$$

$$u(x) = c(S-x) - \rho + hx^+ + px^-, \quad \forall x \leq S \tag{6.3.9}$$

PROOF. From the formula (6.3.2) the convergence of S_α to S is clear. Noting that

$$F(S_\alpha) = \frac{p+c}{p+h} - \frac{c}{\alpha(h+p)},$$

we see that S_α increases with α. Moreover an easy calculation shows that

$$u_\alpha(S_\alpha) = hS_\alpha^+ + pS_\alpha^- + \frac{\alpha}{1-\alpha}[c\bar{D} + hE((S_\alpha - D)^+) + pE((S_\alpha - D)^-)],$$

from which it follows, thanks to the choice of ρ that

$$u_\alpha(S_\alpha) - \frac{\rho}{1-\alpha} \to hS - [c\bar{D} + hE((S-D)^+) + pE((S-D)^-)].$$

Now from Theorem 5.6, we have

$$u'_\alpha(x) = -c + h\mathbb{1}_{x>0} - p\mathbb{1}_{x<0} + z_\alpha(x),$$

where

$$z_\alpha(x) = \sum_{n=1}^{\infty} \alpha^{n-1} g_\alpha \star f^{\star(n-1)}(x),$$

with

$$g_\alpha(x) = (c(1-\alpha) + \alpha h - \alpha(h+p)\bar{F}(x))^+.$$

Again we can see that $z_\alpha(x) \to z(x)$, defined in equation (6.3.7). So we obtain (6.3.6). To prove (6.3.9) we use $z(x) = 0, \forall x \leq S,$ hence

$$u'(x) = -c + h\mathbb{1}_{x>0} - p\mathbb{1}_{x<0}, \forall x \leq S.$$

Integrating between x and S, and using $u(S) = hS - \rho$, formula (6.3.9) follows. To prove (6.3.8), we note that

$$u'(x) = h + Eu'(x-D), \ \forall x > S.$$

Next we verify that $Eu(S-D) = 0$. This follows from (6.3.8), applied at $x = S-D$ and taking the mathematical expectation. So integrating between S and x, we obtain easily (6.3.9).

The proof has been completed. Note that $u \in C^1$, except for $x = 0$. □

6.3.2. PROBABILISTIC INTERPRETATION. We proceed as in the no shortage case. We first interpret the number ρ. Consider one period, starting with an inventory equal to S. At the end of the period the inventory is $S - D$. In order to face the new demand with an inventory equal to S in the next period, we have to order D. At the end of the period we pay for storage $h(S-D)^+$ and if we have shortage we pay $pE(S-D)^-$. Therefore the expected cost on a period is

$$c\bar{D} + hE(S-D)^+ + pE(S-D)^-,$$

which is exactly ρ. To find the optimal S we minimize this expression in S.

Consider now $x \leq S$ and the following sequence

$$\begin{aligned} &y_1 = x, \;\; v_1 = S - x; \\ &y_2 = S - D_1, \;\; v_2 = D_1; \\ &\cdots \\ &y_{n+1} = S - D_n, \;\; q_{n+1} = D_n. \end{aligned}$$

If we discount with rate α we obtain the cost

$$u_\alpha(x) = hx^+ + px^- + c(S-x) + \sum_{n=1}^{\infty} \alpha^n [c\bar{D} + hE(S-D)^+ + pE(S-D)^-],$$

therefore

$$u_\alpha(x) = hx^+ + px^- + c(S-x) + \frac{\alpha\rho}{1-\alpha},$$

and

$$u_\alpha(x) - \frac{\rho}{1-\alpha} \to = hx^+ + px^- + c(S-x) - \rho.$$

Consider now the case $x > S$. We define

$$y_{n+1} = y_n - D_n, \; y_1 = x.$$

Let

$$\tau = \inf\{n \geq 1 | y_n \leq S\}.$$

Starting at period τ we are back in the situation where the stock is below S and we proceed as follows

$$\begin{aligned} &y_\tau = y_\tau, \;\; v_\tau = S - v_\tau, \\ &y_{\tau+n} = S - D_{\tau+n-1}, \;\; v_{\tau+n} = D_{\tau+n-1}. \end{aligned}$$

The corresponding cost is given by

$$\begin{aligned} u_\alpha(x) = E\sum_{n=1}^{\tau-1} &\alpha^{n-1} h y_n + \\ &+ E\alpha^{\tau-1}[-cy_\tau + cS + hy_\tau^+ + py_\tau^- \\ &+ \sum_{n=1}^{\infty} E\alpha^{\tau-1+n}[cD_{\tau+n-1} + h(S - D_{\tau+n-1})^+ + p(S - D_{\tau+n-1})^-] \end{aligned}$$

hence also by appropriate conditioning, taking into account that $D_{\tau+n-1}$ is independent from τ, for $n \geq 1$

$$\begin{aligned} u_\alpha(x) - \frac{\rho}{1-\alpha} &= E\sum_{n=1}^{\tau-1} \alpha^{n-1}(hy_n - \rho) \\ &+ E\alpha^{\tau-1}(-cy_\tau + cS - \rho + hy_\tau^+ + py_\tau^-) \end{aligned}$$

and letting α tend to 1 we obtain

$$\begin{aligned} u_\alpha(x) - \frac{\rho}{1-\alpha} &\to E\sum_{n=1}^{\tau-1}(hy_n - \rho) \\ &+ E(-cy_\tau + cS - \rho + hy_\tau^+ + py_\tau^-), \end{aligned}$$

and the right hand side is the probabilistic interpretation of $u(x)$, as defined by (6.3.8), (6.3.9).

6.4. DETERMINISTIC CASE

The deterministic case can treated as a particular case of the stochastic case. However we can look directly at the formulas obtained in section 5.3.

In the no shortage case we have the formulas

$$u_\alpha(x) = \begin{cases} (h-c)x + \dfrac{cD}{1-\alpha}, & \forall x \leq D \\ hx + \alpha u_\alpha(x-D), & \forall x \geq D \end{cases} \tag{6.4.1}$$

The Base stock $S_\alpha = D$, does not depend on α, so $S = D$, and $\rho = cD$. We obtain that

$$u_\alpha(x) - \frac{cD}{1-\alpha} \to u(x), \tag{6.4.2}$$

with $u(x)$ solution of

$$u(x) = \begin{cases} (h-c)x, & \forall x \leq D \\ hx - cD + u(x-D), & \forall x \geq D \end{cases}$$

Similarly in the backlog case we have the relations

$$u_\alpha(x) = \begin{cases} (h-c)x + \dfrac{cD}{1-\alpha}, & \forall x \leq D \\ hx + \alpha u_\alpha(x-D), & \forall x \geq D \end{cases}$$

so $S = D$, $\rho = cD$. We have again (6.4.2), with

$$u(x) = \begin{cases} hx^+ + px^- - cx, & \forall x \leq D \\ hx^+ + px^- - cD + u(x-D), & \forall x \geq D \end{cases}$$

CHAPTER 7

OPTIMAL STOPPING PROBLEMS

We consider in this chapter a situation where decisions are stopping times. They can be reduced to the general case, but it is preferable to use the specific aspects of this type of problems. We will begin by defining the functional equation of interest, which results from Dynamic Programming, study it directly then obtain the interpretation as a control problem with stopping times as decisions.

7.1. DYNAMIC PROGRAMMING

We present the functional equation of Dynamic Programming, after presenting the control problem of interest.

7.1.1. FORMULATION OF THE PROBLEM. We consider the framework of Chapter 4. We recall that a decision rule is an infinite sequence

$$\hat{V} = \{\hat{v}_1, \cdots, \hat{v}_n, \cdots\},$$

in which

$$\hat{v}_n = \hat{v}_n(x_1, \cdots, x_n),$$

is a Borel map with values in U. From the decision rule we construct the probability $P^{V,x}$ of the canonical stochastic process y_n, given $y_1 = x$, representing the state trajectory. The state space is $X = R^d$. We recall the filtration

$$\mathcal{Y}^n = \sigma(y_1, \cdots, y_n).$$

In the present situation, there will be a new decision variable of a different nature, namely a stopping time τ with respect to the filtration $\mathcal{Y}^n$.

We consider the function $l(x, v)$ such that

$$l(x, v) \geq 0; \quad \text{l.s.c.} \tag{7.1.1}$$

In addition we shall consider a function $\psi(x)$ such that

$$\psi(x) \geq 0; \quad \text{l.s.c.} \tag{7.1.2}$$

The cost objective is defined by

$$J_x(V, \tau) = E^{V,x}\left[\sum_{n=1}^{\tau-1} \alpha^{n-1} l(y_n, v_n) + \alpha^{\tau-1}\psi(y_\tau)\right] \tag{7.1.3}$$

7.1.2. DYNAMIC PROGRAMMING EQUATION. The functional equation giving the value function of the preceding problem is defined by

$$u(x) = \min\{\psi(x), \inf_{v \in U}[l(x,v) + \alpha \Phi^v u(x)]\}, \forall x. \tag{7.1.4}$$

This functional equation combines the nonlinear operator studied in Chapter 4, with the function $\psi(x)$. We call this function, the obstacle. This refers to the fact that the solution $u(x)$ must remain below the obstacle. As we see in equation (7.1.4), it represents in the cost functional the payoff received at the decision time τ.

We shall consider the the ceiling function $w_0(x)$, defined in equation (4.3.4).

We state the main result

Theorem 7.1. *Under the assumptions* (7.1.1)*,* (7.1.2) (4.3.1)*,* (4.3.2) *and* (4.3.6) *then the set of solutions of the functional equation* (7.1.4) *satisfying*

$$0 \leq u(x) \leq w_0(x) \; \forall x,$$

is not empty and has a minimum and a maximum solution, denoted respectively $\underline{u}(x)$ *and* $\overline{u}(x)$*. The minimum solution is l.s.c. and the maximum solution is u.s.c. if the functions* $l(x,v)$ *and* ψ *are continuous.*

PROOF. Let z be l.s.c. and positive. Define

$$Tz(x) = \inf_{v \in U}[l(x,v) + \alpha \Phi^v z(x)].$$

We know from Chapter 4 that $Tz(x)$ is also l.s.c. and positive. Set

$$Sz(x) = \min[\psi(x), Tz(x)].$$

We use the following exercise to claim that $Sz(x)$ is also l.s.c. and positive.

Exercise 7.1. If f,g are l.s.i and bounded below then $\min(f,g)$ is also l.s.c. and bounded below.

The function u solution of equation (7.1.4) is a fixed point of the map S. The map S is increasing, i.e. if $z_1 \leq z_2$ then $Sz_1 \leq Sz_2$. If $z \geq 0$ then $Sz \geq 0$. Suppose $z \leq \bar{w}$ then

$$Sz(x) \leq l(x,v) + \alpha \Phi^{v_0} w_0(x) = w_0(x).$$

We have checked that S maps the interval $[0, w_0]$ into itself. Consider next the decreasing scheme

$$u^{n+1} = Su^n, \;\; u^0 = w_0.$$

Clearly

$$u^0 \geq u^1 \geq \cdots \geq u^n \cdots \geq 0.$$

We can state that $u^n \downarrow u$ where u is a positive Borel function. Clearly $u^{n+1} \geq Su$, hence also $u \geq Su$. Also

$$\begin{aligned} u^{n+1}(x) &\leq \psi(x) \\ u^{n+1}(x) &\leq l(x,v) + \alpha \Phi^v u^n(x), \forall v \end{aligned}$$

Letting n tend to $+\infty$ yields

$$\begin{aligned} u(x) &\leq \psi(x) \\ u(x) &\leq l(x,v) + \alpha \Phi^v u(x), \forall v \end{aligned}$$

Therefore $u \leq Su$. So the limit u is a fixed point of S. Moreover it is the maximum solution in the interval $[0, w_0]$. It is u.s.c. as decreasing limit of a sequence of continuous functions.

Exercise 7.2. Check that u is the maximum sub-solution, namely if $\tilde{u} \leq S\tilde{u}$ and $0 \leq \tilde{u} \leq w_0$, then $\tilde{u} \leq u$.

The maximum solution is denoted by $\bar{u}$.

Proof follow up. We next consider the increasing sequence

$$u_{n+1} = Su_n, \;\; u_0 = 0.$$

Clearly

$$u_0 \leq u_1 \leq \cdots \leq u_n \cdots \leq w_0.$$

Therefore $u_n \uparrow u$ and $0 \leq u \leq w_0$ and u is l.s.c. Since $u_{n+1} \leq Su$ we obtain $u \leq Su$. On the other hand, there exists a Borel map $\hat{v}_n(x)$ such that

$$u_{n+1}(x) = \min[\psi(x), l(x, \hat{v}_n(x)) + \alpha\Phi^{\hat{v}_n(x)}u_n(x)].$$

Let $m \geq n$. One can write

$$u_{m+1}(x) \geq \min[\psi(x), l(x, \hat{v}_m(x)) + \alpha\Phi^{\hat{v}_m(x)}u_n(x)],$$

hence also

$$u(x) \geq \min[\psi(x), l(x, \hat{v}_m(x)) + \alpha\Phi^{\hat{v}_m(x)}u_n(x)].$$

We fix n and let $m \to \infty$. Using the assumption (4.3.2) we see that the sequence $\hat{v}_m(x)$ remains in a compact set (x is fixed, and the compact set depends on x). We can thus extract a converging subsequence $\hat{v}_{m_k}(x) \to v^*(x)$. Since the function of v

$$\min[\psi(x), l(x, v) + \alpha\Phi^v u_n(x)] \text{ is l.s.c.,}$$

we can assert that

$$u(x) \geq \min[\psi(x), l(x, v^*(x)) + \alpha\Phi^{v^*(x)}u_n(x)].$$

Letting next $n \to \infty$ we obtain $u \geq Su$ and from the reverse inequality we see that u is a fixed point of S.

Exercise 7.3. Check that u obtained by the increasing sequence is the minimum solution, namely if $\tilde{u} = S\tilde{u}$ and $0 \leq \tilde{u} \leq w_0$, then $\tilde{u} \geq u$.

The minimum solution is denoted by $\underline{u}(x)$. □

7.2. INTERPRETATION

We now interpret the minimum solution as the value function of the problem defined with the objective function (7.1.3). We however begin by showing that the functional equation (7.1.4) can be written in a standard form, provided that we extend the control.

7.2.1. ANOTHER FORMULATION. It is not hard to show that fixed points u of S can also be written as the solution of an ordinary Dynamic Programming functional equation, provided one introduces an additional control

Exercise 7.4. Show that

$$u(x) = \inf_{\{v \in U, 0 \leq d \leq 1\}} [d\psi(x) + (1-d)l(x, v) + \alpha(1-d)\Phi^v u(x)].$$

In this form, the interpretation of $u(x)$, in fact the interpretation of the minimum solution as the value function of a stochastic control problem becomes standard. We introduce V, D decision rules associated with the control variables v, d, namely

$$\begin{aligned} V &= \{v_1, \cdots, v_n, \cdots\} \\ D &= \{d_1, \cdots, d_n, \cdots\} \end{aligned}$$

and

$$\begin{aligned} v_n &= v_n(x_1, \cdots, x_n) \\ d_n &= d_n(x_1, \cdots, x_n) \end{aligned}$$

The functions v_n and d_n are Borel functions, with values in U and $[0,1]$ respectively. Since Φ^v does not depend on d the probability defined on $\Omega = X^N, \mathcal{A}$ associated to the decision rule V, D and to the initial state x depends only on V and x and is denoted $P^{V,x}$. Considering the canonical process y_n (see Chapter 3) we know that

$$y_1 = x, P^{V,x} \text{ a.s.}$$

The decision rules define stochastic processes $v_n(y_1, \cdots, y_n)$ and $d_n(y_1, \cdots, y_n)$ adapted to the filtration $\mathcal{Y}^n$ generated by the canonical process.

We now define a payoff associated with V, D, x as follows

$$J_x(V, D) = E^{V,x} \sum_{j=1}^{\infty} \alpha^{j-1}(1 - d_{j-1}) \cdots (1 - d_1)[d_j \psi(y_j) + (1 - d_j) l(y_j, v_j)].$$

From the theory of Chapter 4 we can assert that the minimum solution between 0 and w_0 obtained with the increasing sequence is the value function corresponding to this payoff. Namely, we have

$$\underline{u}(x) = \inf_{V,D} J_x(V, D)$$

7.2.2. OPTIMAL STOPPING FORMULATION. To use the trick described in the previous section is not the best way to deal with optimal stopping problems. We proceed now directly with stopping times. Let us consider θ to be a stopping time with respect to the filtration $\mathcal{Y}^n$. We then define the payoff

$$J_x(V, \theta) = E^{V,x} \left[\sum_{j=1}^{\theta-1} \alpha^{j-1} l(y_j, v_j) + \alpha^{\theta-1} \psi(y_\theta) 1\!\!1_{\theta<\infty} \right]$$

Theorem 7.2. *Under the assumptions of Theorem 7.1 the minimum solution $\underline{u}(x)$ satisfies*

$$\underline{u}(x) = \inf_{V,\theta} J_x(V, \theta).$$

PROOF. Let V be a decision rule and θ a stopping time with respect to $\mathcal{Y}^n$. Consider the increasing sequence $u_n(x)$. We can write for $0 \le j \le N-1$

$$\begin{aligned} &\alpha^{N-j-1} u_{j+1}(y_{N-j}) 1\!\!1_{N-j<\theta} \le \\ &\qquad \alpha^{N-j-1} l(y_{N-j}, v_{N-j}) 1\!\!1_{N-j<\theta} + \alpha^{N-j} E^{V,x}[u_j(y_{N-j+1}) 1\!\!1_{N-j<\theta} | \mathcal{Y}^{N-j}] \end{aligned}$$

Taking expectations on both sides yields

$$\begin{aligned} &E^{V,x}(\alpha^{N-j-1} u_{j+1}(y_{N-j}) 1\!\!1_{N-j<\theta}) \le \\ &\qquad E^{V,x}(\alpha^{N-j-1} l(y_{N-j}, v_{N-j}) 1\!\!1_{N-j<\theta}) + E^{V,x}(\alpha^{N-j} u_j(y_{N-j+1}) 1\!\!1_{N-j<\theta}) \end{aligned}$$

We add up for $j = 0\cdots,N-1$. We obtain

$$E^{V,x}\sum_{k=1}^{(\theta-1)\wedge N}\alpha^{k-1}u_{N-k+1}(y_k)\leq \\ E^{V,x}\sum_{k=1}^{(\theta-1)\wedge N}\alpha^{k-1}l(y_k,v_k)+E^{V,x}\sum_{k=2}^{\theta\wedge(N+1)}\alpha^{k-1}u_{N-k+1}(y_k).$$

Canceling terms we deduce

$$u_N(x)E^{V,x}\mathbb{1}_{\theta\geq 2}\leq E^{V,x}\sum_{k=1}^{(\theta-1)\wedge N}\alpha^{k-1}l(y_k,v_k)+E^{V,x}\alpha^{\theta-1}u_{N-\theta+1}(y_\theta)\mathbb{1}_{2\leq\theta<N+1},$$

where we have used the fact that $u_0(x)=0$. By adding on both sides of this relation $u_N(x)E^{V,x}\mathbb{1}_{\theta=1}$, we obtain by letting $N\to\infty$ on the right hand side

$$u^N(x)\leq \inf_{V,\theta}J_x(V,\theta).$$

Therefore also

$$\underline{u}(x)\leq \inf_{V,\theta}J_x(V,\theta).$$

Next define the optimal feedback $\hat{v}(x)$ which achieves the minimum of $l(x,v)+\alpha\Phi^v\underline{u}(x)$ for $v\in U$. We associate to this feedback a decision rule $\hat{V}$. We then define the exit time

$$\widehat{\theta}=\inf\{n\geq 1|\underline{u}(y_n)=\psi(y_n)\},$$

with

$$\widehat{\theta}=\infty \text{ if } \underline{u}(y_n)<\psi(y_n),\forall n.$$

We check that $\forall N$

$$\underline{u}(x)=E^{\hat{V},x}\left[\sum_{k=1}^{(\widehat{\theta}-1)\wedge N}\alpha^{k-1}l(y_k,v_k)+\alpha^{(\widehat{\theta}-1)\wedge N}\underline{u}(y_{\widehat{\theta}\wedge(N+1)})\right],$$

and thus also

$$\underline{u}(x)\geq E^{\hat{V},x}\left[\sum_{k=1}^{(\widehat{\theta}-1)\wedge N}\alpha^{k-1}l(y_k,v_k)+\alpha^{\widehat{\theta}-1}\psi(y_{\widehat{\theta}})\mathbb{1}_{\widehat{\theta}\leq N+1}\right].$$

Letting N tend to ∞ we obtain

$$\underline{u}(x)\geq J_x(\hat{V},\widehat{\theta}),$$

and the proof of the theorem has been completed. □

7.2.3. DIRECT APPROACH. We have shown that there are two interpretations of the minimum solution $\underline{u}(x)$

$$\underline{u}(x)=\inf_{V,\theta}J_x(V,\theta)=\inf_{V,D}J_x(V,D).$$

We can check directly the equality of the two infimum. We first notice that without loss of generality we can restrict d to take only the values 0 or 1. This is because the optimal feedback obtained from Bellman equation, which allows to obtain the optimal decision rule, satisfies this property.

Now, consider any decision rule D which has this property, we define

$$\theta_D=k,\ \text{ if } d_j=0,\ j\leq k-1, d_k=1$$
$$\theta_D=\infty,\ \text{ if } d_j=0,\forall j$$

then one checks that θ_D is a $\mathcal{Y}^n$ stopping time and

$$J_x(V, \theta_D) = J_x(V, D),$$

from which we deduce $\inf J_x(V, D) \geq \inf J_x(V, \theta)$.

On the other hand, if θ is an arbitrary stopping time, we define

$$\begin{aligned} D_\theta &= (d_1^\theta, \cdots, d_j^\theta, \cdots); \\ d_j^\theta &= 0 \text{ if } j \leq \theta - 1; \\ d_j^\theta &= 1 \text{ if } j \geq \theta, \end{aligned}$$

and we check that

$$J_x(V, \theta) = J_x(V, D_\theta),$$

from which it follows that

$$\inf J_x(V, \theta) \geq \inf J_x(V, D),$$

and the equality follows.

7.3. PENALTY APPROXIMATION

In this section, we develop a new idea, very useful in practice. Recall that we used the terminology "obstacle" for the function ψ, reminiscent of the fact that $u \leq \psi$. The idea is that we can waive this constraint by paying a penalty. When this penalty is infinite, the obstacle constraint will be satisfied.

7.3.1. FORMULATION. To simplify a little bit we consider the optimal stopping problem without control. Moreover we assume

$$l, \psi \in C, \;\; l \geq 0, \;\; \psi \geq 0, \tag{7.3.1}$$

where $C = C(X)$ denotes the space of uniformly continuous and bounded functions on X. With this additional assumption there exists a unique solution u of

$$u(x) = \min[\psi(x), l(x) + \alpha \Phi u(x)]. \tag{7.3.2}$$

We know that this solution is continuous. To guarantee that it is uniformly continuous we make an additional assumption on Φ, which strengthens (4.3.1)

$$\Phi\varphi \in C, \text{if } \varphi \text{ is continuous, bounded.} \tag{7.3.3}$$

We associate to the preceding problem the following one

$$u_\epsilon(x) = l(x) + \alpha \Phi u_\epsilon(x) - \frac{1}{\epsilon}(u_\epsilon - \psi)^+(x), \tag{7.3.4}$$

called the penalty approximation. The logic is simple. Instead of imposing the constraint $u \leq \psi$ we accept that our unknown function exceeds the constraint ψ, but then we add to the cost l a penalty term, to be paid only when the function exceeds the constraint. Obviously as ϵ goes to 0 the penalty term becomes very large, forcing the unknown function to fulfill the constraint. Note that the equation (7.3.4) is a non-linear problem.

7.3.2. SOLUTION OF THE PENALTY PROBLEM. We first prove the

Lemma 7.1. *Under the assumptions* (7.3.1) *and* (7.3.3) *there exists a unique solution positive solution in C of the penalty problem* (7.3.4).

PROOF. It is convenient to write the equivalent form of (7.3.4)

$$u_\epsilon\left(1+\frac{1}{\epsilon}\right) = l + \alpha\Phi u_\epsilon + \frac{1}{\epsilon}\min(\psi, u_\epsilon)$$

Exercise 7.5. Show the preceding assertion.

Therefore u_ϵ is a fixed point of the map $w_\epsilon = T_\epsilon(z)$ defined by

$$w_\epsilon\left(1+\frac{1}{\epsilon}\right) = l + \alpha\Phi w_\epsilon + \frac{1}{\epsilon}\min(\psi, z)$$

Exercise 7.6. Check that T_ϵ is a contraction with constant $\dfrac{1}{1+\epsilon(1-\alpha)}$.

This completes the proof of the Lemma. □

Lemma 7.2. *We have*

$$0 \le u_\epsilon \le \max\left(||\psi||, \frac{||l||}{1-\alpha}\right).$$

Moreover u_ϵ decreases as $\epsilon \downarrow 0$.

PROOF. If $z \ge 0$, $T_\epsilon(z) \ge 0$. Hence since T_ϵ is monotone we obtain $u_\epsilon \ge 0$.

Exercise 7.7. Check that for any $z \ge 0$ one has

$$||T_\epsilon z|| \le \max\left(||\psi||, \frac{||l||}{1-\alpha}\right),$$

hence the same estimate follows for the solution u_ϵ.

Follow up: To prove the monotonicity consider $\epsilon' < \epsilon$ and set $w = T_\epsilon(u_{\epsilon'})$. We first verify that

$$(w - u_{\epsilon'})\left(1+\frac{1}{\epsilon}\right) = \alpha\Phi(w - u_{\epsilon'}) + \left(\frac{1}{\epsilon'} - \frac{1}{\epsilon}\right)(u_{\epsilon'} - \psi)^+,$$

from which it follows that $w - u_{\epsilon'} \ge 0$. Therefore $u_{\epsilon'} \le T_\epsilon(u_{\epsilon'})$. Iterating we get $u_{\epsilon'} \le T_\epsilon^n(u_{\epsilon'})$. Letting n tend to ∞ we get $u_{\epsilon'} \le u_\epsilon$. The proof of the lemma has been completed. □

7.3.3. CONVERGENCE. We state the following convergence result.

Theorem 7.3. *Under the assumptions* (7.3.1) *and* (7.3.3) *one has $u_\epsilon \downarrow u$ the solution of the equation* (7.3.2).

PROOF. We can assert that $u_\epsilon \downarrow u^*$. From (7.3.4) one can write

$$u_\epsilon \le l + \alpha\Phi u_\epsilon,$$

hence

$$u^* \le l + \alpha\Phi u^*.$$

Moreover, since u_ϵ is bounded, we can assert that $(u_\epsilon - \psi)^+ \to 0$. Therefore

$$u^* \le \psi.$$

Note that

$$(u_\epsilon - \psi)^-(u_\epsilon - l - \alpha\Phi u_\epsilon) = 0.$$

We can pass to the limit and obtain

$$(u^* - \psi)^-(u^* - l - \alpha\Phi u^*) = 0.$$

Using the property $u^* - \psi \leq 0$, we can also write

$$(u^* - \psi)(u^* - l - \alpha\Phi u^*) = 0,$$

which implies that u^* is a solution of (7.3.2). The proof has been completed. □

7.4. ERGODIC CASE

We study now the ergodic problem, letting $\alpha \to 1$.

7.4.1. SETTING OF THE MODEL. We shall make here the assumptions of Theorem 7.1 and also

$$l, \psi \geq 0, \text{bounded}, \quad l(x) \geq l_0 > 0. \tag{7.4.1}$$

We consider the problem

$$u_\alpha(x) = \min\left\{\psi(x), \inf_{v \in U}[l(x,v) + \alpha\Phi^v u_\alpha(x)]\right\}, \forall x. \tag{7.4.2}$$

We know that it has a unique l.s.c. continuous solution (by contraction mapping arguments).

The issue is to study the limit of u_α as α tends to 1. In fact this problem will be easier than the one without the obstacle ψ. The reason is because the function u_α will remain bounded.

7.4.2. THE MAIN RESULT. We state the main result

Theorem 7.4. *We make the assumptions of Theorem 7.1 and* (7.4.1). *Consider the equation*

$$u(x) = \min\left\{\psi(x), \inf_{v \in U}[l(x,v) + \Phi^v u(x)]\right\}, \forall x, \tag{7.4.3}$$

then it has a positive bounded l.s.c. solution which is unique. Moreover $u_\alpha(x) \uparrow u(x)$ *as* $\alpha \uparrow 1$.

PROOF. We prove the existence and uniqueness of the solution z. We cannot use the argument of contraction mapping, since $\alpha = 1$. To prove the existence of a positive bounded l.s.c. solution, we consider the increasing sequence

$$u_0 = 0$$
$$u_{n+1}(x) = \min\left\{\psi(x), \inf_{v \in U}[l(x,v) + \Phi^v u_n(x)]\right\}$$

then clearly

$$u_0 \leq u_1 \cdots \leq u_n \leq \cdots \psi,$$

hence $u_n \uparrow u \geq 0$. Moreover $u \leq \psi$ and is l.s.c.

Reasoning as in the proof of Theorem 7.1 we check that the limit u has the desired properties and is a solution. Let us prove the uniqueness of a Borel bounded solution. We cannot use the contraction mapping argument. We use the following trick. We first exclude the case $\psi = 0$ in which case the unique solution is $u = 0$.

Suppose there exist two positive bounded solutions z_1, z_2. Suppose there exists a point x^* such that $z_2(x^*) < z_1(x^*)$. Then we have

$$\gamma = \inf_{\{x|z_1(x)>0\}} \frac{z_2(x)}{z_1(x)} < 1.$$

We are going to show a contradiction. We are going to construct a number β with $\gamma < \beta \leq 1$ such that $\beta z_1(x) \leq z_2(x), \forall x$. It will follow

$$\gamma = \inf_{\{x|z_1(x)>0\}} \frac{z_2(x)}{z_1(x)} \geq \beta,$$

which is a contradiction.

Let $0 < \lambda < 1$. A solution z satisfies also

$$z(x) = \min\left\{\lambda\psi(x) + (1-\lambda)z(x), \inf_{v\in U}[\lambda l(x,v) + (1-\lambda)z(x) + \lambda\Phi^v z(x)]\right\}.$$

This is in particular satisfied with z_1 and z_2. note that

$$\beta z_1(x) = \min\left\{\beta\lambda\psi(x) + (1-\lambda)\beta z_1(x), \inf_{v\in U}[\lambda\beta l(x,v) + (1-\lambda)\beta z_1(x) + \lambda\Phi^v \beta z_1(x)]\right\}.$$

One could claim that $\beta z_1(x) \leq z_2(x)$ whenever

$$\begin{aligned} \beta\lambda\psi(x) + (1-\lambda)\beta z_1(x) &\leq \lambda\psi(x) + (1-\lambda)z_2(x) \\ \lambda\beta l(x,v) + (1-\lambda)\beta z_1(x) &\leq \lambda l(x,v) + (1-\lambda)z_2(x) \end{aligned}$$

Since $z_2(x) \geq \gamma z_1(x)$, it is sufficient to guarantee that

$$\begin{aligned} \beta\lambda\psi(x) + (1-\lambda)\beta z_1(x) &\leq \lambda\psi(x) + (1-\lambda)\gamma z_1(x) \\ \lambda\beta l(x,v) + (1-\lambda)\beta z_1(x) &\leq \lambda l(x,v) + (1-\lambda)\gamma z_1(x) \end{aligned}$$

We must have

$$\begin{aligned} (1-\lambda)(\beta-\gamma)z_1(x) &\leq \lambda(1-\beta)\psi(x) \\ (1-\lambda)(\beta-\gamma)z_1(x) &\leq \lambda(1-\beta)l(x,v) \end{aligned}$$

It is sufficient to check that

$$\begin{aligned} (1-\lambda)(\beta-\gamma) &\leq \lambda(1-\beta) \\ (1-\lambda)(\beta-\gamma) &\leq \lambda(1-\beta)\frac{l_0}{||\psi||} \end{aligned}$$

It is sufficient to choose β such that

$$\frac{\beta-\gamma}{1-\beta} = \rho = \frac{\lambda}{1-\lambda}\min\left(1, \frac{l_0}{||\psi||}\right),$$

hence

$$\beta = \frac{\rho+\gamma}{1+\rho},$$

and $\gamma < \beta \leq 1$. The contradiction has been established. From this contradiction it follows that $z_2(x) \geq z_1(x), \forall x$. By symmetry between z_1 and z_2 we obtain $z_1 = z_2$. The uniqueness has then been proven.

To complete the proof, we have to check the convergence property. We first prove that

$$\alpha \leq \alpha' \Rightarrow u_\alpha \leq u_{\alpha'}.$$

This property is obtained by considering the increasing sequence

$$u_{\alpha,n+1}(x) = \min\left[\psi(x), \inf_v (l(x,v) + \alpha\Phi^v u_{\alpha,n}(x)\right],$$

and $u_{\alpha,0}(x) = 0$. It is clear that

$$u_{\alpha,n}(x) \le u_{\alpha',n}(x) \to u_{\alpha,n+1}(x) \le u_{\alpha',n+1}(x),$$

and the monotonicity result follows.

We next write

$$u_\alpha(x) = \min\left\{\psi(x), \inf_{v\in U}[l(x,v) + \Phi^v u_\alpha(x) - (1-\alpha)\Phi^v u_\alpha(x)]\right\}, \forall x.$$

Since $||(1-\alpha)\Phi^v u_\alpha|| \le (1-\alpha)||\psi||$, hence tends to 0, we see easily that $u_\alpha \uparrow u^*$ which a solution of (7.4.3). The proof has been completed. □

Remark. If the functions $l(x,v)$ and ψ are continuous then the solution of (7.4.3) is also continuous. This is checked by defining a monotone decreasing sequence, converging towards a solution, which is u.s.c. By uniqueness, the solution is continuous.

7.4.3. PROBABILISTIC INTERPRETATION. We consider as in section 7.2, a decision rule V and a stopping time θ. We then define the pay-off

$$J_x(V,\theta) = E^{V,x}\left[\sum_{j=1}^{\theta-1} l(y_j,v_j) + \psi(y_\theta)1\!\!1_{\theta<\infty}\right].$$

We can state the

Theorem 7.5. *We make the assumptions of Theorem 7.4. We then have*

$$u(x) = \inf_{V,\theta} J_x(V,\theta).$$

PROOF. We first remark that

$$\psi(x) \ge J_x(V,\theta) \ge l_0 E^{V,x}(\theta - 1),$$

so, without any restriction, we can assume that $\theta < \infty$ a.s. $P^{V,x}$. One then checks the inequality

$$u(x) \le E^{V,x}\left[\sum_{j=1}^{\theta-1} l(y_j,v_j) + u(y_\theta)\right] \le J_x(V,\theta).$$

Define next the feedback $\hat{v}(x)$ Borel function with values in U which attains the infimum in v in equation (7.4.3). We then define

$$\widehat{\theta} = \inf\{n \ge 1 | u(y_n) = \psi(y_n)\}.$$

We can assert that

$$E^{\widehat{V},x}\widehat{\theta} \le \frac{\psi(x)}{l_0}.$$

Indeed, for any N we can write

$$u(x) = E^{\hat{V},x}\left[\sum_{j=1}^{\widehat{\theta}\wedge N-1} l(y_j,v_j) + u(y_{\widehat{\theta}\wedge N})\right] \ge l_0 E^{\hat{V},x}\widehat{\theta}\wedge N.$$

Since $u(x) \leq \psi(x)$ we also obtain

$$E^{\widehat{V},x}\widehat{\theta} \wedge N \leq \frac{\psi(x)}{l_0},$$

and letting N go to ∞ the desired estimate is obtained. We therefore can write

$$u(x) = E^{\hat{V},x}\left[\sum_{j=1}^{\widehat{\theta}-1} l(y_j, v_j) + u(y_{\widehat{\theta}})\right].$$

Also $u(y_{\widehat{\theta}}) = \psi(y_{\widehat{\theta}})$ since $\widehat{\theta} < \infty$ a.s. $P^{\hat{V},x}$. Therefore

$$u(x) = J_x(\hat{V}, \widehat{\theta}),$$

and the proof has been completed. □

Exercise 7.8. Check that when $l_0 = 0$, $u_n \uparrow \underline{u}$ minimum solution of equation (7.4.3). This minimum solution is the value function defined in the statement of Theorem 7.5.

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CHAPTER 8

IMPULSE CONTROL

Impulse Control has been studied mainly in continuous time, see [**8**]. Indeed, in continuous time the difference between an impulse control and a continuous control is quite apparent. An impulse control allows for jumps on the state of the system. A continuous control allows only for a continuous evolution of the state of the system and forbids jumps. In discrete time, there is of course no continuous evolution, so it looks as if the concept loses its specificity; the state at time $n+1$ can be considered as a jump from the state at time n. However, we will identify an analogy with optimal stopping. One can embed optimal stopping into a standard control problem in discrete time, but one can also treat it in a specific way. Similarly, an impulse control problem can be embedded into a standard control problem in discrete time, but one can also consider it in a specific manner. In that case it will look like optimal stopping, but with a sequence of stopping times instead of a single stopping time. We will begin by describing the model. Then we introduce the basic tool, which is the functional equation arising from Dynamic Programming. We study it directly by analytic techniques. We finally proceed with the interpretation.

8.1. DESCRIPTION OF THE MODEL

8.1.1. TRANSITION PROBABILITY. In this set up we will consider a controlled Markov chain with special features. It is at the same time a generalization and a special case of the general framework. However, we shall need the fact that X is a vector space or a subspace of a vector space. To fix the ideas we take $X = R^d$ or $X = (R^d)^+$. We consider as usual a transition probability $\pi(x, v, d\eta)$. We define then the transition probability

$$\pi(x, v, w, d\eta) = \pi(x + w, v, d\eta).$$

It is indeed a generalization of the usual case, which is recovered by fixing the value $w = 0$, but also a particular case, considering the pair v, w as a control.

We associate to the transition probability the operator

$$\Phi^{v,w} f(x) = \int_{R^d} f(\eta)\pi(x, v, w, d\eta).$$

Considering the operator Φ^v associated to $\pi(x, v, d\eta)$ we have obviously

$$\Phi^{v,w} f(x) = \Phi^v f(x + w).$$

The controls v, w are mathematically similar, but in practice act differently. The control v modifies the transition probability, but does not change x. The control w modifies x. Its influence is instantaneous, instead of taking place at the next period. This is why it is called an *impulse*. So the difference remains meaningful, even though we are in discrete time. We remark also that if the assumption (4.3.1) is satisfied for Φ^v, it carries over to $\Phi^{v,w}$.

8.1.2. ASSUMPTIONS. We now define a cost function. Let $l(x,v)$ satisfying (4.3.2), we set

$$l(x,v,w) = l(x+w,v) + c(w). \tag{8.1.1}$$

We consider only $w \in (R^d)^+$ in accordance with the applications we have in mind. We then assume the special form

$$\begin{aligned} &c(w) = K\mathbb{1}_{w\neq 0} + c_0(w) \\ &c_0(w) \text{ is sublinear positive continuous, } \ c_0(0) = 0 \\ &c_0(w) \to \infty \text{ as } |w| \to \infty \end{aligned} \tag{8.1.2}$$

We note that (4.3.2) is satisfied by the function $l(x,v,w)$.

Remark. Of course the important element in the structure of $c(w)$ is the presence of K, which is only there when an action is taken, $w \neq 0$. It is called a *fixed cost* (in the inventory literature it is also called *set up cost).* We note also that, even when $l(x,v)$ is continuous, the function $l(x,v,w)$ will remain l.s.c and not continuous.

We consider a decision rule V,W extending the usual definition for V. We associate a probability $P^{V,W,x}$ on Ω, $\mathcal{A}$ for which $y_1 = x$ a.s. We will consider the pay-off function

$$J_x(V,W) = E^{V,W,x}\left[\sum_{n=1}^{\infty} \alpha^{n-1} l(y_n,v_n,w_n)\right], \tag{8.1.3}$$

and, as usual, we are interested in the value function

$$u(x) = \inf_{V,W} J_x(V,W). \tag{8.1.4}$$

We shall need a ceiling function. We cannot use the symbol $w_0(x)$ for the ceiling function, since we have a new control w_n. By taking $w_n = 0$ and $v_n = v_0$ where v_0 is a fixed element of U, the space of constraints, a closed subset of a metric space. Define

$$\psi_0(x) = \sum_{n=1}^{\infty} \alpha^{n-1}(\Phi^{v_0})^{n-1} l_v(x), \tag{8.1.5}$$

where $l_v(x) = l(x,v)$. We make the assumption

$$\psi_0(x) < \infty, \forall x. \tag{8.1.6}$$

We take ψ_0 as the ceiling function. It is the same as for the control problem, without impulses.

We can now write the Bellman equation corresponding to the pay-off (8.1.3). This is the functional equation

$$u(x) = \inf_{\left\{\begin{smallmatrix} v\in U \\ w\geq 0\end{smallmatrix}\right\}} [l(x,v,w) + \alpha\Phi^{v,w}u(x)]. \tag{8.1.7}$$

The preceding equation is equivalent to

$$u(x) = \inf_{\left\{\begin{smallmatrix} v\in U \\ w\geq 0\end{smallmatrix}\right\}} [l(x+w,v) + c(w) + \alpha\Phi^{v}u(x+w)] \tag{8.1.8}$$

8.2. STUDY OF THE FUNCTIONAL EQUATION

8.2.1. GENERAL RESULT. As a consequence of Theorem 4.3 we can state the following

Theorem 8.1. *Under the assumptions* (4.3.1), (4.3.2), (8.1.2), (8.1.6) *the set of solutions of the functional equation* (8.1.8) *satisfying*

$$0 \le u(x) \le \psi_0(x) \forall x,$$

is not empty and has a minimum and a maximum solution, denoted respectively $\underline{u}(x)$ *and* $\overline{u}(x)$*. The minimum solution is l.s.c.*

As a consequence of Theorem 4.4 we can also assert that

Theorem 8.2. *Under the assumptions of Theorem 8.1 the minimum solution is the value function*

$$\underline{u}(x) = \inf_{V,W} J_x(V, W).$$

Moreover there exist optimal decision rules $\hat{V}$, $\hat{W}$ *obtained through an optimal feedback* $\hat{v}(x), \hat{w}(x)$.

8.2.2. THE BOUNDED CASE. In this section we assume in addition that

$$l(x, v) \text{ is bounded,} \tag{8.2.1}$$

then we have

Theorem 8.3. *Under the assumptions of Theorem 8.1 and* (8.2.1)*, the solution* u *of* (8.1.8) *in the interval* $0, \psi_0$ *is unique.*

PROOF. The function ψ_0 is bounded. So all solutions between the minimum and the maximum are bounded. We can then use a contraction mapping argument. Indeed, let z be a l.s.c bounded function. We consider the map

$$Tz(x) = \inf_{\left\{ \begin{array}{l} v \in U \\ w \ge 0 \end{array} \right\}} [l(x + w, v) + c(w) + \alpha \Phi^v z(x + w)].$$

It is easy to check that it is a contraction with coefficient α. Hence the uniqueness is obtained. □

8.3. ANOTHER FORMULATION

8.3.1. REWRITING BELLMAN EQUATION. In this section we consider the bounded case, hence we have uniqueness of the solution u. We can rewrite the equation (8.1.8) as follows

$$\begin{aligned} u(x) = \min \Bigg[&\inf_{v \in U} (l(x, v) + \alpha \Phi^v u(x)), \\ &K + \inf_{\left\{ \begin{array}{l} v \in U \\ w \ge 0, \ne 0 \end{array} \right\}} (l(x + w, v) + c_0(w) + \alpha \Phi^v u(x + w)) \Bigg]. \end{aligned} \tag{8.3.1}$$

We will now consider another formulation of (8.3.1).

8.3.2. SEPARATING VARIABLES. In this section, we make a stronger assumption on the function $l(x, v)$. We shall assume that

(8.3.2) $\quad l(x, v)$ is uniformly continuous and bounded, positive.

We shall look for functions $u(x)$ in $C = C(X)$ the space of functions uniformly continuous and bounded.

We define two operators on C, namely

$$T(z)(x) = \inf_{v \in U}[l(x, v) + \alpha \Phi^v z(x)],$$

and

$$M(z)(x) = K + \inf_{w \geq 0 \neq 0}(c_0(w) + z(x + w)).$$

We consider the following problem: find $u \in C$ with

$$u(x) = \min[M(u)(x), T(u)(x)]. \tag{8.3.3}$$

Our objective is to prove the following

Theorem 8.4. *Under the assumptions* (4.3.1)*,* (4.3.2)*,* (8.1.2)*,* (8.3.2) *then there exists one and only one positive, bounded solution of the functional equation* (8.3.3)*. This solution belongs to* C*.*

8.3.3. IDENTITY OF THE TWO FORMULATIONS. It is not obvious that problems (8.3.3) and (8.3.1) which is equivalent to (8.1.8) coincide. Indeed the solution of (8.1.8) is also the solution of

$$u(x) = \min[M(T(u))(x), T(u)(x)], \tag{8.3.4}$$

which is not (8.3.3). We are going to prove that solutions of (8.3.3) are also solutions of (8.3.4), hence of (8.1.8). Since the solution of (8.1.8) is unique, it is sufficient to prove that the solutions (8.3.3) exist. There is then only one solution of (8.3.3), and the two problems are equivalent. To prove that the solutions of (8.3.3) are also solutions of (8.3.4) we have to prove that if u is a solution (8.3.3), then

$$u(x) = M(u)(x) \Rightarrow u(x) = M(T(u))(x), \tag{8.3.5}$$

instead of just $u(x) \leq M(T(u))(x)$. Indeed, from

$$u \leq Tu \Rightarrow M(u) \leq M(T(u)),$$

we get

$$u(x) \leq \min[M(T(u))(x), T(u)(x)],$$

and (8.3.5) implies also the reverse inequality. So proving (8.3.5) is the only thing which remains.

Let x such that $u(x) = M(u)(x)$. Let $\epsilon < \frac{K}{2}$ we can find w_x such that

$$u(x) \geq K + c_0(w_x) + u(x + w_x) - \epsilon.$$

We are going to show that

$$u(x + w_x) = T(u)(x + w_x), \tag{8.3.6}$$

which implies

$$u(x) \geq K + c_0(w_x) + T(u)(x + w_x) - \epsilon,$$

hence

$$u(x) \geq M(T(u))(x) - \epsilon.$$

Since ϵ is arbitrarily small we have $u(x) \geq M(T(u))(x)$ Since the reverse inequality is true, we have the equality, hence the u is also a solution of (8.3.4).

To prove (8.3.6) we claim that if (8.3.6) is not true, then one should have

$$u(x + w_x) = M(u)(x + w_x).$$

But then there will exists $\tilde{w}_x$ such that

$$u(x + w_x) \geq K + c_0(\tilde{w}_x) + u(x + w_x + \tilde{w}_x) - \epsilon.$$

Collecting results we can write

$$u(x) \geq 2K + c_0(w_x) + c_0(\tilde{w}_x) + u(x + w_x + \tilde{w}_x) - 2\epsilon.$$

Using the sub-linearity of c_0 we can assert

$$u(x) \geq 2K + c_0(w_x + \tilde{w}_x) + u(x + w_x + \tilde{w}_x) - 2\epsilon.$$

Therefore also

$$u(x) > K + \inf_{w \geq 0}[c_0(w) + u(x + w)],$$

by the choice of ϵ. This is a contradiction.

Remark 8.1. We can then define an optimal feedback $\hat{v}(x)$, $\hat{w}(x)$ separately. Define first $\hat{v}(x)$ by minimizing $l(x, v) + \alpha\Phi^v u(x)$. Next if $u(x) < M(u)(x)$ we have $\hat{w}(x) = 0$. If $u(x) = M(u)(x)$ then $\hat{w}(x)$ minimizes $c_0(w) + u(x + w)$ in $w \geq 0$.

8.3.4. DIRECT STUDY OF THE SECOND FORMULATION. If we want to study the fixed point of (8.3.3), the equivalence obtained above and the general theory applied to the pair v, w shows that the fixed point exists and is unique. So the proof of Theorem 8.4 can be done in this way. We give a direct proof here, which allows to introduce useful techniques, similar to those used for optimal stopping. In fact, the problem of impulse control we are considering here corresponds to choosing an optimal sequence of stopping times instead of just one.

We begin by proving that there is at most one positive, bounded solution. We use a method reminiscent of the one used for the ergodic optimal stopping problem, see section 7.4.2. However, one has to be careful about a big difference. We need here $\alpha < 1$ to work with bounded solutions, whereas in the optimal stopping problem there was a natural bound and we could take $\alpha = 1$.

PROOF. *Uniqueness*

Consider now two solutions u_1, u_2 which are positive and bounded by a constant C_0. Define

$$\gamma = \sup\{\beta \in [0, 1] | \beta u_1(x) \leq u_2(x), \forall x\}.$$

This number is well defined. Indeed

$$\gamma = \inf_{\{x|u_1(x)>0\}} \frac{u_2(x)}{u_1(x)}.$$

We are going to show that $\gamma = 1$. This will imply $u_1(x) \leq u_2(x), \forall x$. By symmetry, we will get also the reverse property, hence the uniqueness. □

Assume $0 \leq \gamma < 1$. We are going to construct a number β such that $\gamma < \beta \leq 1$ such that

$$\beta u_1(x) \leq u_2(x), \forall x,$$

which will imply a contradiction. Now βu_1 satisfies

$$\beta u_1 = \min[\beta M(u_1), \beta T(u_1)].$$

Now $\beta T(u_1)(x) \leq T(\beta u_1)(x)$. Suppose we find β such that $\gamma < \beta \leq 1$ and

$$\beta M(u_1)(x) \leq M(u_2)(x),$$

then it will follow from Lemma 8.1 below that $\beta u_1(x) \leq u_2(x), \forall x$ hence $\beta \leq \gamma$, which will be a contradiction. Checking the desired property means

$$K\beta + \inf_{w\geq 0}(\beta c_0(w) + \beta u_1(x+w)) \leq K + \inf_{w\geq 0}(c_0(w) + u_2(x+w)).$$

Using $\gamma u_1 \leq u_2$ it is enough to verify

$$K\beta + \inf_{w\geq 0}(\beta c_0(w) + \beta u_1(x+w)) \leq K + \inf_{w\geq 0}(c_0(w) + \gamma u_1(x+w)),$$

or

$$K\beta + \inf_{w\geq 0}(c_0(w) + \beta u_1(x+w)) \leq K + \inf_{w\geq 0}(c_0(w) + \gamma u_1(x+w)).$$

We want $\beta > \gamma$. Since $u_1 - C_0 \leq 0$, $u_2 - C_0 \leq 0$ we have also

$$\inf_{w\geq 0}(c_0(w) + \beta u_1(x+w)) \leq \inf_{w\geq 0}(c_0(w) + \gamma u_1(x+w)) + (\beta-\gamma)C_0.$$

Hence it is sufficient to choose β such that

$$(\beta-\gamma)C_0 \leq K(1-\beta).$$

Setting

$$\rho = \frac{K}{C_0}.$$

We can take

$$\beta = \frac{\rho+\gamma}{\rho+1}, \;\; \gamma < \beta \leq 1.$$

We now prove the existence. We begin with some preliminary results.

Lemma 8.1. *Let $\psi \in C$ and $S(\psi) = z \in C$ solution of*

$$z = \min[\psi, T(z)],$$

then if $\zeta \in C$ satisfies

$$\zeta \leq \psi, \;\; \zeta \leq T(\zeta),$$

we have $\zeta \leq z$.

PROOF. Consider the map $\Sigma : C \to C$ defined by

$$\Sigma(z) = \min[\psi, T(z)].$$

The operator Σ is a contraction and is monotone increasing. Since from the assumption

$$z = \Sigma^n(z), \;\; \zeta \leq \Sigma^n(\zeta),$$

hence

$$z - \zeta \geq \Sigma^n(z) - \Sigma^n(\zeta).$$

Letting n tend to ∞, the right hand side tends to 0. Hence the result has been obtained. □

Lemma 8.2. *The operator S is concave, namely*

$$S(\lambda\psi_1 + (1-\lambda)\psi_1) \geq \lambda S(\psi_1) + (1-\lambda)S(\psi_2), \forall \lambda \in [0,1]$$

Exercise 8.1. Prove that T is concave and check then that S is also concave, by making use of Lemma 8.1.

Lemma 8.3. *The operator S is monotone increasing*

Exercise 8.2. Prove Lemma 8.3 by making use of Lemma 8.1.

Define next

$$A(z)(x) = S(M)(z)(x).$$

Exercise 8.3. Check that A is concave, monotone increasing.

Let $\overline{u}$ to be the unique solution of $\overline{u} = T(\overline{u})$. Then $0 \leq \overline{u} \leq C_0$. We can define $\lambda > 0$ such that $\lambda\overline{u}(x) \leq K, \forall x$.

We state the following

Lemma 8.4. *Let $z, \zeta \in C$ positive, and $\beta \in [0,1]$ such that*

$$z(x) - \zeta(x) \leq \beta z(x), \forall x, \tag{8.3.7}$$

then

$$A(z)(x) - A(\zeta)(x) \leq \beta(1-\lambda)A(z)(x), \forall x. \tag{8.3.8}$$

We have, by assumption, $(1-\beta)z(x) \leq \zeta(x)$. From the monotonicity and the concavity of A we can write

$$\begin{aligned} A(\zeta)(x) &\geq A((1-\beta)(z))(x) \\ &\geq (1-\beta)A(z)(x) + \beta A(0)(x) \end{aligned}$$

However $A(0) = S(M(0)) = S(K)$. But from $\lambda\overline{u}(x) \leq K$ and $\lambda\overline{u}(x) = \lambda T(\overline{u})(x)$ hence $\lambda\overline{u}(x) \leq T(\lambda\overline{u})(x)$ we deduce, using Lemma 8.1

$$\lambda\overline{u}(x) \leq S(K) = A(0).$$

Collecting results we write

$$A(\zeta)(x) \geq (1-\beta)A(z)(x) + \beta\lambda\overline{u}(x).$$

Now

$$\begin{aligned} &A(z) \leq TA(z) \\ &\overline{u} = T\overline{u} \end{aligned}$$

and by a reasoning similar to that of Lemma 8.1 we deduce $A(z)(x) \leq \overline{u}(x)$. Therefore

$$A(\zeta)(x) \geq (1-\beta)A(z)(x) + \beta\lambda A(z)(x),$$

and this is exactly the property (8.3.8).

We can can turn to the direct proof of existence in Theorem 8.4.

Proof. *Existence:*

This amounts to proving that the operator A has a fixed point. We consider the decreasing sequence

$$\begin{aligned} &u^{n+1} = A(u^n) \\ &u^0 = \overline{u} \end{aligned}.$$

We have

$$u^0 \geq \cdots \geq u^n \geq \cdots.$$

Since $u^0 - u^1 \leq u^0$, we deduce from Lemma 8.4

$$A(u_0) - A(u_1) \leq (1-\lambda)A(u_0),$$

which reads

$$u_1 - u_2 \leq (1-\lambda)u_1.$$

Iterating we get

$$u^m - u^{m+1} \leq (1-\lambda)^m u^m \leq (1-\lambda)^m u^0.$$

We deduce easily, that for $n \geq m$

$$u^m - u^n \leq \frac{(1-\lambda)^m}{\lambda} u^0.$$

Letting n tend to ∞ we get $u_n \downarrow u$, hence also

$$u^m - u \leq \frac{(1-\lambda)^m}{\lambda} u^0 \leq \frac{(1-\lambda)^m}{\lambda} C_0,$$

which proves that the limit $u \in C(X)$. We next check that

$$M(u_n) \downarrow M(u) \text{ and in } C$$
$$T(u_{n+1}) \downarrow T(u) \text{ and in } C$$

Since

$$u_{n+1} = \min[M(u_n), T(u_{n+1})],$$

the limit u is a solution. The proof has been completed. □

8.4. PROBABILISTIC INTERPRETATION

In Theorem 8.2 we have already given a probabilistic interpretation of the solution of the functional equation (8.1.8) in terms of the value function of a stochastic control problem. There are no stopping times in this formulation. In a way similar to what has been seen for optimal stopping, see Theorem 7.5, we will obtain a different formulation involving decision variables which are stopping times. In fact, we are going to interpret the second formulation, equation (8.3.3). Here also, to simplify technicalities we limit ourselves to (8.3.3), which means that the cost $l(x, v)$ is bounded.

We take a control policy $V = \{v_1, \cdots, v_n, \cdots\}$ where v_n is $\mathcal{Y}_n$ measurable, and belongs to U. The control policy W will be defined in a slightly different way. We consider a sequence of $\mathcal{Y}^n$ stopping times, $\{\tau_1, \cdots, \tau_n, \cdots\}$. We assume no accumulation $\tau_n < \tau_{n+1}, \forall n$, a.s. Consider the σ- algebra $\mathcal{Y}^{\tau_n}$ of events prior to τ_n and a sequence ξ_{τ_n} of random variables which are $\mathcal{Y}^{\tau_n}$ measurable, with values in $X = R^d$,$\xi_{\tau_n} \geq 0, \neq 0$. We associate the control $W = \{w_1, \cdots, w_n, \cdots\}$ defined as follows

$$w_{\tau_n} = \xi_{\tau_n}$$
$$w_j = 0 \text{ if } j \neq \tau_1, \cdots, \tau_n, \cdots$$

We can check that w_j is $\mathcal{Y}^j$ measurable, $\forall j$. We can then consider $J_x(V, W)$ and write it as

$$J_x(V, W) = E^{V,W,x} \sum_{n=1}^{\infty} \alpha^{\tau_n - 1} \left[K + c_0(\xi_{\tau_n}) + l(y_{\tau_n} + \xi_{\tau_n}, v_{\tau_n}) + \sum_{j=\tau_n+1}^{\tau_{n+1}-1} \alpha^{j-\tau_n} l(y_j, v_j) \right].$$

To the optimal feedback $\hat{v}(x), \hat{w}(x)$ we associate a control policy $\hat{V}, \hat{W}$ defined as follows

$$\hat{v}_n = \hat{v}(y_n)$$
$$\hat{w}_n = \hat{w}(y_n)$$

We next define

$$\begin{aligned}\hat{\tau}_1 &= \inf\{n \geq 1 | \hat{w}_n \neq 0\} \\ \hat{\tau}_2 &= \inf\{n \geq \hat{\tau}_1 + 1 | \hat{w}_n \neq 0\} \\ \cdots\end{aligned}$$

and

$$\hat{\xi}_{\hat{\tau}_j} = \hat{w}_{\hat{\tau}_j}.$$

Using arguments similar to those of Theorem 7.2, we can obtain

$$u(x) = J_x(\hat{V}, \hat{W}) = \inf J_x(V, W)$$

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CHAPTER 9

INVENTORY CONTROL WITH SET UP COST

The techniques of impulse control will now be applied to study the situation of set up costs in inventory control. The Base stock policy introduced in Chapter 5 will be replaced by the concept of s, S policy introduced by H. Scarf [**35**] and studied by many authors. We will first consider a deterministic situation without backlog, but different from that of section 5.3 of Chapter 5.

9.1. DETERMINISTIC MODEL

9.1.1. DESCRIPTION OF THE MODEL. We assume that the inventory evolves as follows

$$y_{n+1} = y_n + v_n - D \quad y_1 = x, \tag{9.1.1}$$

with the constraints

$$v_n \geq 0, \; y_n \geq 0. \tag{9.1.2}$$

The novelty is the state constraint. There is no lost demand. Since the demand D is deterministic, this translates into a constraint on the control, which depends on the state

$$v_n \geq D - y_n. \tag{9.1.3}$$

Define $V = \{v_1, \cdots, v_n, \cdots\}$ we consider the objective functional

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1}[K 1\!\!1_{v_n>0} + cv_n + hy_n], \tag{9.1.4}$$

and we want to study the value function

$$u(x) = \inf_V J_x(V). \tag{9.1.5}$$

The new term is the fixed cost $K 1\!\!1_{v_n>0}$ to be paid only when the order v_n is not 0. This is not a continuous function. If we denote

$$l(x, v) = K 1\!\!1_{v>0} + cv + hx, \tag{9.1.6}$$

then the function $l(x, v)$ is not continuous in v. However it is l.s.c.

9.1.2. BELLMAN EQUATION. The Bellman equation is written as follows

$$u(x) = hx + \inf_{v \geq (D-x)^+}[K 1\!\!1_{v>0} + cv + \alpha u(x + v - D)]. \tag{9.1.7}$$

This functional equation belongs to the general category studied in Chapter 4. We will focus on the specific aspects.

A first important fact is the following

Lemma 9.1. *The set of v in the right hand side of equation* (9.1.7) *can be bounded by*

$$cv \leq \frac{K+cD}{1-\alpha} + \frac{\alpha Dh}{(1-\alpha)^2}. \tag{9.1.8}$$

PROOF. The result will follow from the estimate

$$u(x) \geq \frac{hx}{1-\alpha} - \frac{\alpha Dh}{(1-\alpha)^2},$$

which is obtained by iterating on the functional equation (9.1.7), starting with $u(x) \geq hx$. Next, we look for a ceiling function for $u(x)$. Because of the constraint, we cannot as usual the function corresponding to the choice $v = 0$. We shall take $v = D$. So we consider the functional equation

$$w_0(x) = hx + K + cD + \alpha w_0(x),$$

which is simply

$$w_0(x) = \frac{hx+K+cD}{1-\alpha}. \tag{9.1.9}$$

We look for solutions of (9.1.7) which are in the interval $0 \leq u(x) \leq w_o(x)$. From the bound below of u and the bound above provided by the ceiling function, we deduce from the equation (9.1.7), that the values of v can be restricted by the constraint

$$hx + cv + \frac{\alpha h(x-D)}{1-\alpha} - \frac{hD\alpha^2}{(1-\alpha)^2} \leq \frac{hx+K+cD}{1-\alpha}.$$

We see that the term in x drops out, and there remains the estimate (9.1.8). This concludes the proof. □

Since the set of v is bounded, we can apply the techniques of Theorem (4.2.4) to claim

Theorem 9.1. *Under the assumption* (9.1.6), *the functional equation* (9.1.7) *has one and only one solution which is l.s.c. This solution is the value function* (9.1.5). *Moreover there exists an optimal control policy $\hat{V}$ defined by a feedback $\hat{v}(x)$, which achieves the infimum in the right hand side of* (9.1.7).

9.1.3. OBTAINING THE OPTIMAL CONTROL POLICY. In this section we are interested in characterizing the optimal feedback $\hat{v}(x)$. We first give a different, more convenient form of the equation (9.1.7). We introduce

$$z(x) = cx + \alpha u(x). \tag{9.1.10}$$

Then it is fairly easy to rewrite equation (9.1.7) as follows

$$u(x) = (h-c)x + cD + \inf_{\eta \geq \max(x,D)} [K\mathbb{1}_{\eta > x} + z(\eta - D)]. \tag{9.1.11}$$

We begin with the inequality

Lemma 9.2.

$$z(x) \leq K + z(y), \forall y > x. \tag{9.1.12}$$

PROOF. Let $x < y$. From the equation (9.1.11) we can write

$$\begin{aligned} u(x)+(c-h)x &= cD+\inf_{\eta\geq \max(x,D)}[K\mathbb{1}_{\eta>x}+z(\eta-D)] \\ &\leq cD+K+\inf_{\eta\geq \max(y,D)} z(\eta-D) \\ &\leq K+cD+\inf_{\eta\geq \max(y,D)}[K\mathbb{1}_{\eta>y}+z(\eta-D)] \\ &\leq K+u(y)+(c-h)y, \end{aligned}$$

which implies

$$u(x)+cx \leq K+u(y)+cy.$$

But then

$$\begin{aligned} z(x) &= \alpha(u(x)+cx)+cx(1-\alpha) \\ &\leq \alpha K+\alpha(u(y)+cy)+cy(1-\alpha) \\ &\leq K+z(y), \end{aligned}$$

and the result follows. □

We then deduce

Proposition 9.1. *The solution $u(x)$ satisfies the property*

$$x<D \Longrightarrow u(x)=(h-c)x+cD+K+\inf\nolimits_{\eta\geq 0} z(\eta) \tag{9.1.13}$$

$$x\geq D \Longrightarrow u(x)=hx+\alpha u(x-D). \tag{9.1.14}$$

PROOF. For $x < D$, the equation (9.1.11) writes

$$u(x)=(h-c)x+cD+\inf_{\eta\geq D}[K\mathbb{1}_{\eta>x}+z(\eta-D)],$$

and thus

$$u(x)=(h-c)x+cD+K+\inf_{\eta\geq D} z(\eta-D),$$

and the property (9.1.13) follows. Next if $x \geq D$, the equation (9.1.11) becomes

$$u(x)=(h-c)x+cD+\inf_{\eta\geq x}[K\mathbb{1}_{\eta>x}+z(\eta-D)].$$

But from Lemma 9.2 we have

$$z(x-D)\leq K+z(\eta-D), \forall x<\eta,$$

hence

$$u(x)=(h-c)x+cD+z(x-D).$$

Recalling the definition of $z(x)$ we deduce (9.1.15). This concludes the proof. □

Lemma 9.3. *We have the property*

$$\alpha u(0)\leq u(0)-cD.$$

PROOF. We have $\alpha u(0) = z(0)$. Now from (9.1.13)

$$u(0)-cD=K+\inf_{\eta\geq 0} z(\eta).$$

But $\inf_{\eta\geq 0} z(\eta)$ is attained at a point $\eta^* \geq 0$. But from Lemma 9.2, we have

$$z(0)\leq K+z(\eta^*),$$

and the result follows.

We can now assert the main result □

Theorem 9.2. *We assume* (9.1.6). *The optimal feedback is characterized as follows*

$$\hat{v}(x) = \begin{cases} 0, & \text{if } x \geq D \\ D(1+\hat{k}) - x, & \text{if } x < D \end{cases} \tag{9.1.15}$$

where $\hat{k}$ *is a positive integer which minimizes the quantity*

$$\lambda(k) = \frac{K + c(k+1)D + \alpha h D F(k)}{1 - \alpha^{k+1}}, \tag{9.1.16}$$

where

$$F(k) = \frac{\alpha^{k+1} - (k+1)\alpha + k}{(1-\alpha)^2}. \tag{9.1.17}$$

Proof. From Proposition 9.1 it follows that $u(x)$ is piece wise linear, but not continuous. The discontinuity points are $Dk, k = 1, 2, \cdots$ Let us write

$$u(0) = cD + K + \inf_{\eta \geq 0} z(\eta).$$

We can easily obtain formulas for $u((kD)^+)$ and $u((kD)^-)$, where these symbols mean the limit to the right and the limit to the left at point kD. Indeed for $k \geq 1$

$$u(((j+1)D)^-) = hD(j+1) + \alpha u((jD)^-),$$

and we deduce easily

$$u(((k+1)D)^-) = hD \sum_{j=0}^{k} (k+1-j)\alpha^j + \alpha^k (u(0) - cD), \ k \geq 0. \tag{9.1.18}$$

Similarly for $k \geq 0$

$$u(((j+1)D)^+) = hD(j+1) + \alpha u((jD)^+),$$

and thus

$$u(((k+1)D)^+) = hD \sum_{j=0}^{k} (k+1-j)\alpha^j + \alpha^{k+1} u(0), \ k \geq 0. \tag{9.1.19}$$

From Lemma 9.3 we can see that

$$u(((k+1)D)^+) \leq u(((k+1)D)^-), \forall k \geq 0$$

Since $z(x)$ is also piece-wise linear, with the same discontinuity points as $u(x)$, it follows from this inequality that

$$\inf_{\eta \geq 0} z(\eta) = \inf_{k \geq 0} \left[ckD + \alpha u((kD)^+)\right]. \tag{9.1.20}$$

We can compute for $k \geq 1$

$$\begin{aligned} F(k) &= \sum_{j=0}^{k-1} (k-j)\alpha^j \\ &= k\frac{1-\alpha^k}{1-\alpha} - \frac{\alpha}{(1-\alpha)^2}(1 - k\alpha^{k-1} + (k-1)\alpha^k) \\ &= \frac{\alpha^{k+1} - (k+1)\alpha + k}{(1-\alpha)^2}. \end{aligned}$$

Note that this expression is also valid for $k = 0$, by taking $F(0) = 0$. So we can write

$$u\left((kD)^+\right) = F(k) + \alpha^k u(0), k \geq 0$$

Therefore from (9.1.20), we can state

$$\inf_{\eta \geq 0} z(\eta) = \inf_{k \geq 0} \left[ckD + \alpha hDF(k) + \alpha^{k+1}u(0)\right]. \tag{9.1.21}$$

Now from (9.1.13), we have

$$u(0) = K + \inf_{k \geq 0} \left[c(k+1)D + \alpha hDF(k) + \alpha^{k+1}u(0)\right],$$

from which one gets easily

$$u(0) = \inf_{k \geq 0} \lambda(k),$$

with $\lambda(k)$ given by (9.1.16). If $\hat{k}$ achieves the infimum, then $\hat{k}D$ achieves also the infimum of $z(\eta)$. Since for $x < D$, this infimum is achieved by $x + \hat{v}(x) - D$, from the definition of the feedback, we can state that

$$x + \hat{v}(x) - D = \hat{k}D,$$

and we obtain (9.1.15). Note that $\lambda(k)$ attains its infimum since $\lambda(k) \to \infty$ as $k \to \infty$. □

We now consider particular cases

Proposition 9.2. *Assume that*

$$\alpha(K + cD) \leq cD + \alpha hD, \tag{9.1.22}$$

then $\hat{k} = 0$.

Proof. We use the fact that $\hat{k}$ is also the value of k for which $c(k+1)D + \alpha hDF(k) + \alpha^{k+1}u(0)$ attains its minimum. Let us prove that this function is increasing in k, when the assumption is made. Using the fact that

$$F(k+1) - F(k) = \frac{1 - \alpha^{k+1}}{1 - \alpha},$$

we must check that for any k, we have

$$u(0)(\alpha^{k+1} - \alpha^{k+2}) \leq cD + \alpha hD\frac{1 - \alpha^{k+1}}{1 - \alpha}.$$

We know that

$$u(0) \leq w(0) = \frac{K + cD}{1 - \alpha}.$$

Therefore, it is sufficient to verify

$$\alpha^{k+1}(K + cD) \leq cD + \alpha hD\frac{1 - \alpha^{k+1}}{1 - \alpha},$$

hence

$$\alpha^{k+1}\left[K + cD + \frac{hD\alpha}{1 - \alpha}\right] \leq cD + \frac{\alpha hD}{1 - \alpha},$$

and it suffices to check that it is true for $k = 0$, which is exactly the condition (9.1.22). □

Remark 9.1. The optimal feedback is said to be derived from an s, S policy, with $s = D$ and $S = D(\hat{k}+1)$. Note that if $K = 0$, then (9.1.22) is satisfied and $\hat{k} = 0$, which implies $s = S$. We have in this case a Base stock policy, with Base stock equal to the demand. If we consider the optimal policy $\hat{V} = \{\hat{v}_1, \cdots, \hat{v}_n, \cdots\}$ and the corresponding evolution of the inventory $\hat{y}_1, \cdots, \hat{y}_n, \cdots$ then we have the property

$$\hat{y}_n \hat{v}_n = 0, \forall n \tag{9.1.23}$$

This property is known as Wagner-Whitin Theorem.

9.2. INVENTORY CONTROL WITH FIXED COST AND NO SHORTAGE

9.2.1. BELLMAN EQUATION. We consider the analogue of section 5.1 of Chapter 5. We define the function $l(x, v)$ as follows

$$l(x, v) = K\mathbb{1}_{v>0} + cv + hx + pE(x + v - D)^{-}, \tag{9.2.1}$$

which is the same as equation (5.1.8) with a fixed cost K whenever an order is made, whatever the size. Note that $K\mathbb{1}_{v>0}$ is l.s.c. It incorporates an ordering cost (purely a variable cost), a holding cost and a penalty for lost sales. We note the inequalities

$$K\mathbb{1}_{v>0} + cv + hx \leq l(x, v) \leq K\mathbb{1}_{v>0} + cv + hx + p\bar{D}, \tag{9.2.2}$$

where $\bar{D} = ED$. Therefore the function $l(x, v)$ satisfies the properties (4.3.2), (4.5.1) and the first condition (4.5.4), with

$$l_0 = p\bar{D}. \tag{9.2.3}$$

The operator Φ^v is defined by (5.1.2), or equivalently by (5.1.5). We note that assumptions (4.3.1), (4.5.2), (4.5.4) are satisfied.

So the results of Chapter 4, in particular Theorem 4.8 apply. The functional equation (4.3.7) reads

$$u(x) = \inf_{v \geq 0} \{l(x, v) + \alpha E u((x + v - D)^{+})\}. \tag{9.2.4}$$

We can also write it as

$$u(x) = (h - c)x + \inf_{\eta \geq x} \{K\mathbb{1}_{\eta > x} + c\eta + pE((\eta - D)^{-}) + \alpha E u((\eta - D)^{+})\}. \tag{9.2.5}$$

The corresponding control problem is formulated as follows: The state equations are

$$y_{n+1} = (y_n + v_n - D_n)^{+} \quad y_1 = x, \tag{9.2.6}$$

where the D_n are independent and identically distributed, with probability density $f(.)$ and CDF $F(.)$. The objective functional is

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1} E l(y_n, v_n), \tag{9.2.7}$$

where as usual, $V = \{v_1, \cdots, v_n, \cdots\}$. This formulation is, of course equivalent to that of the controlled Markov chain described in Chapter 4. We can state

Theorem 9.3. *Under the assumptions* (9.2.1) *and* (5.1.2)*, there exists one and only one solution of the equation* (9.2.4) *in the space* B_1*. This function is l.s.c. There exists an optimal feedback* $\hat{v}(x)$ *with the estimate*

$$\hat{v}(x) \leq \frac{p\bar{D}}{c(1-\alpha)}. \tag{9.2.8}$$

The solution $u(x) = \inf_V J_x(V)$*, the value function. There exists an optimal control policy* $\hat{V}$*, derived from the feedback* $\hat{v}(x)$*.*

9.2.2. OPTIMAL POLICY. The important question is to characterize the optimal feedback, as we did in the case without fixed cost $K = 0$, see Chapter 5, section 5.1.3. When there is a fixed cost, we cannot get a Base stock policy. We have already seen in the deterministic case, see section 9.1, that the optimal feedback is obtained from an s, S policy, with $s = D$. We shall see that this structure remains valid in the stochastic case, with naturally different values of s, S. Following the theory of impulse control developed in Chapter 8, we will determine the values of s, S.

We introduce the operators

$$T(u)(x) = hx + pE((x-D)^-) + \alpha E u((x-D)^+), \tag{9.2.9}$$

$$M(u)(x) = K + \inf_{\eta \geq x}[(c-h)(\eta - x) + u(\eta)]. \tag{9.2.10}$$

We have now a situation similar to that of section. 8.3. We note that the functional equation (9.2.5) is equivalent to

$$u(x) = \min[T(u)(x), M(T(u))(x)].$$

However, this is also equivalent to

$$u(x) = \min[T(u)(x), M(u)(x)]. \tag{9.2.11}$$

This formulation is the impulse control formulation of (9.2.5). There is no "continuous" control, so the operator $T(u)(x)$ is affine. We are in the conditions of Theorem 8.4, with the important difference concerning the boundedness of the costs. Note that the variable ordering cost is here $(c-h)v$ so we shall assume

$$p > c > h. \tag{9.2.12}$$

The first inequality is essential, since otherwise there is no incentive to order. The second one is less essential but will be made in coherence with our general formulation of impulse control problems. Moreover it will be essential to obtain an s, S policy. We cannot use Theorem 8.4, to prove the existence and uniqueness of the solution, since the function $l(x, v)$ is not bounded. However, by the equivalence with the formulation (9.2.5), we know that it is the case. It is therefore sufficient to construct a solution for which the optimal feedback is given by an s, S policy.

We begin with some notation. Define

$$g(x) = (p - \alpha(c-h))E(x-D)^+ - (p-c)x + p\bar{D}. \tag{9.2.13}$$

For any $s \geq 0$, define next

$$g_s(x) = (g(x) - g(s))\mathbb{1}_{x \geq s}. \tag{9.2.14}$$

Let us set

$$H_s(x) = \sum_{n=1}^{\infty} \alpha^{n-1} g_s \star f^{\star(n-1)}(x), \tag{9.2.15}$$

with the notation of (5.1.17).

Lemma 9.4. *The function $H_s(x)$ attains its minimum in points larger or equal to s. Taking the smallest minimum one defines a function $S(s)$ and*

$$H_s(S(s)) = \inf_{\eta \geq s} H_s(\eta). \tag{9.2.16}$$

Proof. We can write $H_s(x)$ as

$$H_s(x) = g_s(x) + \alpha g_s \star f + \sum_{n=2}^{\infty} \alpha^n g_s \star f^{\star(n)}(x).$$

Consider

$$F^{(n)}(x) = \int_0^x f^{\star(n)}(\xi)\, d\xi,$$

then $F^{(0)}(x) = 1$,$F^{(1)}(x) = F(x)$. If

$$\Gamma(x) = \sum_{n=0}^{\infty} \alpha^n F^{(n)}(x),$$

then we can easily verify the formula

$$H_s(x) = \int_s^x \Gamma(x-\xi)\mu(\xi) d\xi, \tag{9.2.17}$$

with $\mu = g'$. It is useful to give a probabilistic interpretation of the function $F^{(n)}(x)$. In fact, we have

$$F^{(n)}(x) = P(D_1 + \cdots + D_n \leq x),$$

in which $D_1, \cdots, D_n$ are n independent identically distributed random variables with the same probability density $f(x)$. In this way, it is clear that

$$F^{(n)}(x) \uparrow 1, \text{as } x \to \infty, \qquad \Gamma(x) \uparrow \frac{1}{1-\alpha}, \text{as } x \uparrow \infty.$$

Therefore $H_s(x) \to +\infty$ as $x \uparrow \infty$. We can check that

$$H_s'(s+0) = \mu(s) = c(1-\alpha) + \alpha h - (p - \alpha(c-h))\bar{F}(s).$$

Since $p > c$ the function $\mu(s)$ increases from $-(p-c)$ to $c(1-\alpha) + \alpha h$. Therefore it vanishes in a point $\bar{s}$ and is strictly negative for $0 \leq s < \bar{s}$. This $\bar{s}$ is simply the Base stock identified by formula (5.1.14).

Consider first $s \in [0, \bar{s})$. Therefore $H_s'(s+0) < 0$ and the function $H_s(x)$ becomes strictly negative for $x > s$ close to s. Moreover

$$\begin{aligned} H_s(x) &= \int_s^{\bar{s}} \Gamma(x-\xi)\mu(\xi) d\xi + \int_{\bar{s}}^x \Gamma(x-\xi)\mu(\xi) d\xi \\ &\geq \frac{1}{1-\alpha}(g(\bar{s}) - g(s)) + g(x) - g(\bar{s}). \end{aligned}$$

Therefore $H_s(x)$ goes to ∞ as x goes to ∞. Since it is a continuous function, it attains its minimum on $[0, \infty)$. By taking the smallest minimum, if there are several of them, we define uniquely the number $S(s)$. Note also, that if $s \geq \bar{s}$ $H_s(x)$ is increasing in x, and for $s < \bar{s}$ $H_s(x)$ is decreasing in x on the interval $[s, \bar{s})$, hence

$$S(s) = s \text{ if } s \geq \bar{s}, \quad S(s) > \bar{s} \text{ if } s < \bar{s}.$$

It follows also that $H_s'(S(s)) = 0$ for $s < \bar{s}$. Therefore

$$\frac{dH_s(S(s))}{ds} = \frac{\partial}{\partial s} H_s(x)_{|x=S(s)}, \text{ if } s < \bar{s}.$$

So

$$\frac{dH_s(S(s))}{ds} = -\mu(s)\Gamma(S(s) - s) > 0, \text{ if } s < \bar{s}. \tag{9.2.18}$$

Clearly $H_s(S(s)) = 0$ for $s \geq \bar{s}$. The property (9.2.16) has been proven. □

Consider now $S_0 = S(0)$. We have

$$H_0(x) = \int_0^x \Gamma(\xi)\mu(x - \xi)d\xi.$$

So

$$\begin{aligned} H_0'(x) &= \mu(0)\Gamma(x) + \int_0^x \mu'(x - \xi)\Gamma(\xi)\, d\xi \\ &= (c - p)\Gamma(x) + (p - \alpha(c - h)) \int_0^x f(x - \xi)\Gamma(\xi)\, d\xi. \end{aligned}$$

However it is easy to check that $\Gamma(x)$ satisfies the renewal equation

$$\Gamma(x) = 1 + \alpha \int_0^x f(x - \xi)\Gamma(\xi)\, d\xi. \tag{9.2.19}$$

Therefore

$$H_0'(x) = \frac{1}{\alpha}[(p - \alpha(p - h))\Gamma(x) - (p - \alpha(c - h)].$$

Therefore S_0 is given by

$$\Gamma(S_0) = \frac{p - \alpha(c - h)}{p - \alpha(p - h)}. \tag{9.2.20}$$

We shall need the assumption

$$H_0(S_0) = \int_0^{S_0} \Gamma(\xi)\mu(S_0 - \xi)d\xi < -K. \tag{9.2.21}$$

We can then state the following

Theorem 9.4. *Under the assumptions of Theorem 9.3 and* (9.2.12), (9.2.21), *the optimal feedback* $\hat{v}(x)$ *of the functional equation* (9.2.5) *is given by an* s, S *policy. The value of* s *is the unique value such that*

$$K + \inf_{\eta \geq s} H_s(\eta) = 0, \tag{9.2.22}$$

and $S = S(s)$. *Moreover the function* $u(x)$ *is continuous*

PROOF. Consider the function $\inf_{\eta \geq s} H_s(\eta) = H_s(S(s))$.

We have $H_s(S(s)) = 0$ for $s \geq \bar{s}$. From the assumption we have $H_0(S(0)) < -K$. From (9.2.18) we know that $H_s(S(s))$ is strictly increasing in s, for $s < \bar{s}$. So there will be one and only one s, for which (9.2.22) is verified. So the pair s, S is well identified. We are going to construct a solution of (9.2.11), for which the optimal feedback is defined by an s, S policy, with the values s, S just defined. To simplify notation we drop the indices s, S. We use the notation $H(x)$ for $H_s(x)$, with the particular s. Consider the function

$$u(x) = H(x) + (h-c)x + \frac{g(s)}{1-\alpha}. \tag{9.2.23}$$

Since $H(x)$ is continuous, the function $u(x)$ is continuous.

Then $u(x)$ satisfies

$$u(x) = \begin{cases} (h-c)x + \dfrac{g(s)}{1-\alpha}, & x \leq s \\ hx + pE(x-D)^- + \alpha E u((x-D)^+), & x \geq s \end{cases} \tag{9.2.24}$$

and also, from the choice of s

$$K + \inf_{\eta \geq s} [u(\eta) + (c-h)\eta] = \frac{g(s)}{1-\alpha}, \tag{9.2.25}$$

hence also

$$\begin{aligned} u(s) + (c-h)s &= K + \inf_{\eta \geq s} [u(\eta) + (c-h)\eta] \\ &= K + u(S) + (c-h)S \end{aligned} \tag{9.2.26}$$

Since

$$u(x) = -(c-h)x + \frac{g(s)}{1-\alpha}, \quad x \leq s,$$

the function $u(x) + (c-h)x$ is constant on $0, s$. Therefore

$$\inf_{\eta \geq x} [(c-h)\eta + u(\eta)] = \inf_{\eta \geq s} [(c-h)\eta + u(\eta)], \forall x \leq s.$$

Hence the function u satisfies

$$\begin{aligned} u(x) &= M(u)(x), \forall x \leq s \\ u(x) &= T(u)(x), \forall x \geq s \end{aligned}$$

So it remains to prove that

$$\begin{aligned} u(x) &\leq T(u)(x), \forall x \leq s \\ u(x) &\leq M(u)(x), \forall x \geq s \end{aligned}$$

In terms of H the first inequality amounts to $g_s(x) \geq 0, \forall x \leq s$. This is true since $\mu(\xi) < 0, \forall x \leq \xi \leq s < \bar{s}$. The second inequality is more involved. In terms of H it amounts to

$$H(x) \leq K + \inf_{\eta \geq x} H(\eta), \forall x \geq s. \tag{9.2.27}$$

To prove (9.2.27) we first notice that for $x \geq s$ we have $\inf_{\eta \geq x} H(\eta) \geq \inf_{\eta \geq s} H(\eta) = -K$. So the inequality is certainly true whenever $H(x) \leq 0$. Consider then the smallest number called x_0 for which H becomes positive. We have then

$$H(x_0) = 0, \; H(x) < 0 \quad \forall x, s < x < x_0.$$

We can claim that $g_s(x_0) > 0$. This follows from the relation $g_s(x_0) = -\alpha E H(x_0 - D)$ and the properties of x_0 resulting from its definition. To proceed we note that the relation (9.2.15) implies

$$H(x) = g_s(x) + \alpha E H(x-D),$$

which holds for any x, provided that we extend by 0 the definition of H whenever $x \leq 0$.

Let $\xi > 0$. We shall consider the domain $x \geq x_0 - \xi$. Therefore values lower than s are possible and even negative values are possible. We can write the inequality

$$H(x) \leq g_s(x) + \alpha E(H(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}), \forall x. \tag{9.2.28}$$

This follows from the fact that

$$E(H(x-D)\mathbb{1}_{x-D \leq x_0 - \xi}) \leq 0.$$

Define next

$$M(x) = H(x+\xi) + K.$$

Clearly $M(x) \geq 0$. We can write thanks to this positivity

$$M(x) \geq \alpha E(M(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}) + M(x) - \alpha EM(x-D).$$

Now for $x \geq x_0 - \xi$ we have

$$H(x+\xi) - \alpha EH(x+\xi-D) = g_s(x+\xi),$$

hence

$$M(x) - \alpha EM(x-D) = g_s(x+\xi) + K(1-\alpha).$$

Therefore we have

$$M(x) \geq g_s(x+\xi) + \alpha E(M(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}), \forall x \geq x_0 - \xi. \tag{9.2.29}$$

We claim that

$$g_s(x+\xi) \geq g_s(x), \forall x \geq x_0 - \xi. \tag{9.2.30}$$

Indeed, note that $g_s(x+\xi) \geq g_s(x_0) \geq 0$ since g is increasing on $[\bar{s}, \infty)$. So the inequality is obvious, whenever $x < s$. If $x \geq s$ the inequality amounts to $g(x+\xi) \geq g(x)$. This is true whenever $x \geq \bar{s}$, by the monotonicity property. So it remains the case $s < x < \bar{s}$. But in this case $g_s(x) < 0$ and the inequality is satisfied.

Let us define $Y(x) = H(x) - M(x)$. From the inequalities (9.2.29) and (9.2.30) we can state that

$$Y(x) \leq \alpha E(Y(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}), \forall x \geq x_0 - \xi,$$

and $Y(x_0 - \xi) \leq -K$.

These two properties imply $Y(x) \leq 0$, $\forall x \geq x_0 - \xi$ Therefore we have obtained (9.2.27) $\forall x \geq x_0 - \xi$ and in particular for $x > x_0$ as desired. We have completed the proof of the fact that the function $u(x)$ characterized by the function $H(x)$ and defined by (9.2.23) is a solution of (9.2.11) or (9.2.5). So it the value function.

Moreover to the s, S policy is associated a feedback

$$\hat{v}(x) = \begin{cases} S - x, & \text{if } x \leq s \\ 0, & \text{if } x > s \end{cases} \tag{9.2.31}$$

and it is an optimal feedback corresponding to u, which means

$$u(x) = l(x, \hat{v}(x)) + \alpha Eu((x + \hat{v}(x) - D)^+).$$

To this feedback rule we can associate a probability $P^{\hat{V},x}$ for which the canonical process y_n evolves as

$$y_{n+1} = (y_n + \hat{v}_n - D_n)^+,$$

with $\hat{v}_n = \hat{v}(y_n)$. We can also state

$$u(x) = \sum_{j=1}^{n} E^{\hat{V},x}\alpha^{j-1}l(y_j, \hat{v}_j) + \alpha^n E^{\hat{V},x}u(y_{n+1}),$$

for any n. But by definition of the s, S policy, one has $0 \leq y_{n+1} \leq \max(x, S)$. Therefore $u(y_{n+1})$ is bounded, hence as $n \to \infty$ we obtain $u(x) = J_x(\hat{V})$. □

Remark. When $K = 0$, we have $s = S$, and S is given by formula (5.1.14). The function $u(x)$ becomes C^1.

Let us prove the following property

Proposition 9.3. *The function $S(s)$ is decreasing in s, hence $\bar{s} \leq S(s)$ $\leq \min(S_0, \dfrac{p\bar{D}}{c(1-\alpha)})$.*

PROOF. From the relation $H_s'(S(s)) = 0$ we deduce

$$\frac{\partial}{\partial s}H_s'(S(s)) + H_s"(S(s))S'(s) = 0.$$

Now, since $S(s)$ is a minimum $H_s"(S(s)) > 0$. Next use

$$H_s'(x) = \mu(x) + \int_s^x \Gamma'(x-\xi)\mu(\xi)\, d\xi,$$

and thus

$$\frac{\partial}{\partial s}H_s'(x) = -\Gamma'(x-s)\mu(s),$$

which is positive for $s < \bar{s}$, since $\mu(s) < 0$. It follows that $S'(s) < 0$. The result follows. □

Remark. We know that $\bar{s} \leq \dfrac{p\bar{D}}{c(1-\alpha)}$, see Remark 5.1, since $\bar{s}$ corresponds to the Base stock policy in the case without fixed cost. The inequality in the statement implies that $\bar{s} \leq S_0$. Let us prove this fact directly. Indeed we use (from the equation of Γ, (9.2.19))

$$\Gamma(S_0) \leq \frac{1}{1-\alpha F(S_0)} = \frac{1}{1-\alpha+\alpha\bar{F}(S_0)}.$$

It follows that

$$\bar{F}(S_0) \leq \frac{c(1-\alpha)+\alpha h}{p-\alpha(c-h)} = \bar{F}(\bar{s}),$$

from which we deduce $S_0 \geq \bar{s}$. On the other hand S_0 is not necessarily comparable to $\dfrac{p\bar{D}}{c(1-\alpha)}$.

9.2.3. K-CONVEXITY THEORY. When $K = 0$, we also proved that the solution was convex. The Base stock property of the optimal feedback was an easy consequence of the convexity, the continuity and the fact that the minimum solution tends to ∞ as $x \uparrow \infty$. The properties on u were a consequence of similar properties of the increasing sequence u_n.

In the present context the minimum solution cannot be convex. Fortunately this property can be replaced by another one, the K-convexity. This will be sufficient, together with continuity and growth to ∞ at ∞ to guarantee that the optimal feedback is described by an s, S policy. This is the way the s, S policy is proven in the literature, since the seminal work of H. Scarf, [**35**]. We have seen in the previous section that the techniques of impulse control allow to obtain the result, without using K-convexity. L. Benkherouf, [**3**] has also considered a similar approach in the case of backlog, which will be considered in the next section.

There is however an additional difficulty, due to the fact that the function is only defined on positive arguments. So we have to clarify the properties of K-convex functions defined only on positive arguments, which is not usually the case in the literature.

A function f is K convex whenever, for $a < b < c$ we have

$$K + f(c) - f(b) \geq \frac{c-b}{b-a}(f(b) - f(a)).$$

We need to consider functions defined on $[0, \infty)$. In that case the property holds for $a \geq 0$.

Theorem 9.5. *Let $f(x)$ be a K convex function defined on $[0, \infty)$, which is continuous and $f(x) \to \infty$ as $x \to \infty$. Then, there exist numbers s, S with $s \leq S$ and*

$$\begin{aligned}
&f(S) = \min f(x) \\
&s = 0 \text{ if } f(0) \leq K + f(S) \\
&f(s) = K + f(S), \text{ if } f(0) > K + f(S) \\
&f(x) \text{ is strictly decreasing on } [0, s], \text{ if } s > 0 \\
&f(x) \leq K + f(y), \forall s < x < y
\end{aligned}$$

Proof. The continuity and the growth condition imply the existence of S at which the infimum is attained. If the minimum is attained in several points, we take S to be the smallest minimum. Assume first that $f(0) \leq K + f(S)$, then we have to prove only the last statement. Take any x such that $0 < x < S$. We shall prove that $f(x) \leq K + f(S)$. This will imply the result for $x < S$. The statement is obvious whenever $f(x) \leq f(0)$. So we may assume $f(x) > f(0)$. If the statement is not true, then we will have $f(x) > K + f(S)$.

However K-convexity will imply

$$K + f(S) - f(x) \geq \frac{S-x}{x}(f(x) - f(0)),$$

and this leads to a contradiction. It remains to consider the case $y > x > S$ then from K-convexity we get

$$K + f(y) - f(x) \geq \frac{y-x}{x-S}(f(x) - f(S)) \geq 0.$$

The case $s = 0$ has been completed. We may thus assume that $f(0) > K + f(S)$. Thanks to continuity, there exists s, satisfying the condition of its definition. If they are many, we take the smallest value. Let $0 < x < s$, applying K convexity we have

$$0 = K + f(S) - f(s) \geq \frac{S-s}{s-x}(f(s) - f(x)),$$

hence

$$f(x) > f(s), \forall x < s.$$

Now take $0 < x_1 < x_2 < s$, by K convexity again we have

$$K + f(S) - f(x_2) \geq \frac{S - x_2}{x_2 - x_1}(f(x_2) - f(x_1)),$$

and since the left hand side is negative, we get $f(x_2) - f(x_1) < 0$.

Let now $s < x < S$ we have

$$f(s) - f(x) = K + f(S) - f(x) \geq \frac{S - x}{x - s}(f(x) - f(s)),$$

which implies $f(s) > f(x)$. Finally, for $S < x < y$ we can write

$$K + f(y) - f(x) \geq \frac{y - x}{x - S}(f(x) - f(S)) \geq 0.$$

The proof has been completed. □

Consider

$$h(x) = \inf_{\eta \geq x} [f(\eta) + K 1\!\!\mathrm{I}_{\{\eta > x\}}],$$

then

Exercise 9.1. Show that

$$h(x) = \begin{cases} f(x), & \text{if } s = 0 \\ f(s) = K + f(S), & \text{if } 0 < x \leq s \ , \\ f(x), & \text{if } x \geq s \end{cases}$$

and that $h(x)$ is K convex and continuous. Note that $h(x) = f(\max(s, x))$.

Exercise 9.2.

Show that when $s > 0$

$$h(x) \leq h(y) + K, \forall x < y$$

We apply this theory to the functional equation (9.2.5). We set

$$G(\eta) = c\eta + pE((\eta - D)^-) + \alpha E u((\eta - D)^+),$$

then equation (9.2.5) can be written as

$$u(x) = (h - c)x + \inf_{\eta \geq x} \{K 1\!\!\mathrm{I}_{\eta > x} + G(\eta)\}. \tag{9.2.32}$$

We shall prove that

$$u(x), G(x) \quad \text{are } K - \text{convex}.$$

Clearly $G(x) \uparrow \infty$ as $x \uparrow \infty$. In order to guarantee that $G(x)$ is continuous, we shall make the technical assumption

$$f \text{ is continuous.} \tag{9.2.33}$$

Exercise 9.3. Show that whenever (9.2.33) is satisfied, then $Eu((y - D)^+)$ is continuous, hence $G(y)$ is also continuous. Since $G(x)$ is K-convex, continuous and tends to ∞ as x tends to ∞, the optimal feedback is described by an s, S policy.

Theorem 9.6. *Under the assumptions of Theorem 9.3 and* (9.2.12), (9.2.21), (9.2.33) *the solution of the functional equation* (9.2.5) *is continuous, K-convex and tends to ∞ as $x \uparrow \infty$. The optimal feedback is defined by an s, S policy.*

PROOF. The only thing which has to be proven is the K-convexity. It is sufficient to prove the K-convexity of the increasing sequence This is proved by induction. We note that

$$u_{n+1}(x) = (h-c)x + \inf_{\eta \geq x} \{K 1\!\!1_{\eta > x} + G_n(\eta)\},$$

with

$$G_n(\eta) = c\eta + pE((\eta - D)^-) + \alpha E u_n((\eta - D)^+).$$

We prove by induction that the functions $u_n(x), G_n(x)$ are K-convex. Assume that they are continuous and $\to +\infty$ as $x \to +\infty$. From Theorem 9.5, we associate with $G_n(x)$ two numbers s_n, S_n. We have the representation

$$u_{n+1}(x) = (h-c)x + G_n(\max(x, s_n)).$$

From exercise 9.1 we deduce that u_{n+1} is also K-convex and continuous. Also $u_{n+1}(x) \to +\infty$, as $x \to +\infty$. Then we have

$$G_{n+1}(x) = cx + pE((x-D)^-) + \alpha(h-c)E((x-D)^+) + \alpha E G_n(\max(x-D, s_n)).$$

The function

$$cx + pE((x-D)^-) + \alpha(h-c)E((x-D)^+) = (c(1-\alpha) + \alpha h)x + \alpha(h-c)\bar{D}$$
$$+(p + \alpha(h-c))E((x-D)^-),$$

is convex, and $G_n(\max(x-D, s_n))$ is K-convex. It follows that $G_{n+1}(x)$ is K-convex. It is continuous and $\to +\infty$ as $x \to +\infty$.

This property is preserved, when going to the limit. □

9.2.4. PROBABILISTIC APPROACH. We are going to evaluate directly the cost corresponding to an arbitrary s, S policy. We have the following

Theorem 9.7. *Let $V_{s,S}$ be the control corresponding to an s, S policy, defined by the feedback*

$$V_{s,S} = \begin{cases} S - x, & \text{if } x \leq s \\ 0, & \text{if } x > s \end{cases} \tag{9.2.34}$$

The corresponding cost is: For $x \leq s$

$$J_x(V_{s,S}) = (h-c)x + \frac{g(s)}{1-\alpha} + \frac{K + H_s(S)}{(1-\alpha)\Gamma(S-s)}. \tag{9.2.35}$$

For $x > s$, one has

$$J_x(V_{s,S}) = (h-c)x + H_s(x) + \frac{g(s)}{1-\alpha} + (1-(1-\alpha)\Gamma(x-s))\frac{K + H_s(S)}{(1-\alpha)\Gamma(S-s)}. \tag{9.2.36}$$

This function is continuous except for $x = s$.

PROOF. Let $V_{s,S} = (v_1, \cdots, v_n \cdots)$ and $y_1, y_2, \cdots, y_n \cdots$ be the corresponding evolution of stocks. We have $y_1 = x$. We begin by considering the case

I- $x \leq s$**:**

We then have $v_1 = S - x$. We set $\tau_1 = 1$ and we define the sequence of stopping times

$$\tau_{k+1} = \tau_k + 1 + \inf\{n \geq 0 \,|\, D_{\tau_k} + \cdots + D_{\tau_k + n} \geq S - s\}. \tag{9.2.37}$$

Recalling the filtration $\mathcal{F}^n = \sigma(D_1, \cdots, D_n), n \geq 1$ We have the

Lemma 9.5. *The random times τ_k are stopping times with respect to the filtration $\mathcal{F}^{n-1}$. Moreover $\tau_{k+1}-\tau_k-1$ and $D_{\tau_k},\cdots,D_{\tau_k+n}, n\geq 0$, are mutually independent of $\mathcal{F}^{\tau_k}$.*

PROOF. We shall prove that

$$\{\tau_k=n\}\in\mathcal{F}^{n-1}, n\geq 1.$$

We prove it by induction on k. The result is true for $k=1$. Note that $\tau_k\geq k$. Assume that the result is true for τ_k, we want to prove it for τ_{k+1}. It is sufficient to assume $n\geq k+1$, since otherwise the set is empty. If $n=k+1$ we have

$$\{\tau_{k+1}=k+1\}=\{\tau_k=k\}\cap\{D_k\geq S-s\}\in\mathcal{F}^k,$$

and the result is true. So we may assume $n\geq k+2$. We then have

$$\begin{aligned}\{\tau_{k+1}=n\}&=(\{\tau_k=n-1\}\cap\{D_{n-1}\geq S-s\})\\ &\quad\cup_{j=k}^{n-2}\left\{\{\tau_k=j\}\cap\{D_j+\cdots+D_{n-1}\geq S-s\}\cap\{D_j+\cdots+D_{n-2}<S-s\}\right.\\ &\qquad\left.\cap\{D_j+\cdots+D_{n-2}<S-s\}\right\},\end{aligned}$$

which belongs to $\mathcal{F}^{n-1}$, using the induction hypothesis. Let us now prove the second part. Let ξ be a random variable which is $\mathcal{F}^{\tau_k}$ measurable. So $\xi 1\!\!1_{\{\tau_k\leq p\}}$ is $\mathcal{F}^{p-1}$ measurable, $p\geq k$. Consider next a random variable defined by the formula

$$\Phi(\tau_{k+1}-\tau_k-1,D_{\tau_k},\cdots,D_{\tau_k+m}),$$

where the function Φ is deterministic. We must prove

$$E\Phi(\tau_{k+1}-\tau_k-1,D_{\tau_k},\cdots,D_{\tau_k+m})\xi=E\Phi(\tau_{k+1}-\tau_k-1,D_{\tau_k},\cdots,D_{\tau_k+m})E\xi. \tag{9.2.38}$$

Call $X=\Phi(\tau_{k+1}-\tau_k-1,D_{\tau_k},\cdots,D_{\tau_k+m})$. We have

$$\begin{aligned}EX\xi&=\sum_{j=k}^{\infty}\sum_{h=0}^{\infty}E\Phi(h,D_j,\cdots,D_{j+m})1\!\!1_{\tau_k=j}1\!\!1_{\tau_{k+1}=j+h+1}\xi\\ &=\sum_{j=k}^{\infty}\sum_{h=1}^{\infty}E\Phi(h,D_j,\cdots,D_{j+m})1\!\!1_{\tau_k=j}\xi 1\!\!1_{D_j+\cdots+D_{j+h}\geq S-s}1\!\!1_{D_j+\cdots+D_{j+h-1}<S-s}\\ &\quad+\sum_{j=k}^{\infty}E\Phi(0,D_j,\cdots,D_{j+m})1\!\!1_{\tau_k=j}\xi 1\!\!1_{D_j\geq S-s}.\end{aligned}$$

Since $1\!\!1_{\tau_k=j}\xi$ is $\mathcal{F}^{j-1}$ measurable, we obtain

$$\begin{aligned}EX\xi&=\sum_{j=k}^{\infty}\sum_{h=1}^{\infty}E\Phi(h,D_j,\cdots,D_{j+m})1\!\!1_{D_j+\cdots+D_{j+h}\geq S-s}1\!\!1_{D_j+\cdots+D_{j+h-1}<S-s}E1\!\!1_{\tau_k=j}\xi\\ &\quad+\sum_{j=k}^{\infty}E\Phi(0,D_j,\cdots,D_{j+m})1\!\!1_{D_j\geq S-s}E1\!\!1_{\tau_k=j}\xi.\end{aligned}$$

But

$$E\Phi(h,D_j,\cdots,D_{j+m})1\!\!1_{D_j+\cdots+D_{j+h}\geq S-s}=E\Phi(h,D_1,\cdots,D_{1+m})1\!\!1_{D_1+\cdots+D_{1+h}\geq S-s}=A_h;$$

$$E\Phi(0,D_j,\cdots,D_{j+m})1\!\!1_{D_j\geq S-s}=E\Phi(0,D_1,\cdots,D_{1+m})1\!\!1_{D_1\geq S-s}=A_0,$$

independent of j. Therefore

$$EX\xi = \sum_{h=0}^{\infty} A_h E\xi.$$

Moreover, applying this relation with $\xi = 1$, we have also $EX = \sum_{h=0}^{\infty} A_h$ and thus $EX\xi = EX\ E\xi$, which is the property (9.2.38). This concludes the proof of the Lemma. □

We pursue the proof of Theorem 9.7. The ordering times of the control $V_{s,S}$ are the times τ_k. The inventory is then

$$y_{\tau_k} = (S - (D_{\tau_{k-1}} + \cdots + D_{\tau_k - 1}))^+, \forall k \geq 2$$

and the order is

$$v_{\tau_k} = S - y_{\tau_k}, \forall k \geq 2$$

We can next define the lost sales. At time $\tau_k - 1$, one has

$$LS_{\tau_k - 1} = (S - (D_{\tau_{k-1}} + \cdots + D_{\tau_k - 1}))^-, \forall k \geq 2$$

We finally define the inventory between the times $\tau_k + 1$ and $\tau_{k+1} - 1$, if $\tau_{k+1} - 2 \geq \tau_k$. If $\tau_{k+1} - 1 = \tau_k$, we have naturally $y_{\tau_k + 1} = y_{\tau_{k+1}}$. Otherwise

$$\begin{aligned} y_{\tau_k+1} &= S - D_{\tau_k}; \\ &\cdots \\ y_{\tau_{k+1}-1} &= S - (D_{\tau_k} + \cdots + D_{\tau_{k+1}-2}). \end{aligned}$$

Therefore we can write

$$\begin{aligned} (9.2.39)\qquad J_x(V_{s,S}) &= (h-c)x + K + cS + p\sum_{k=2}^{\infty} E\alpha^{\tau_k - 2} LS_{\tau_k - 1} \\ &+ \sum_{k=2}^{\infty} E\alpha^{\tau_k - 1}(K + cS + (h-c)y_{\tau_k}) \\ &+ h\sum_{k=1}^{\infty} E\alpha^{\tau_k - 1}(\alpha y_{\tau_k+1} + \cdots + y_{\tau_k + \tau_{k+1} - \tau_k - 1}\alpha^{\tau_{k+1} - \tau_k - 1}). \\ &= I + II + III \end{aligned}$$

We begin by considering the term III. We first note that

$$\begin{aligned} (9.2.40)\qquad & y_{\tau_k+1} + \cdots + y_{\tau_k + \tau_{k+1} - \tau_k - 1}\alpha^{\tau_{k+1} - \tau_k - 2} \\ &= S\frac{1 - \alpha^{\tau_{k+1} - \tau_k - 1}}{1 - \alpha} - \sum_{j=0}^{\tau_{k+1} - \tau_k - 2} \alpha^j \sum_{l=0}^{j} D_{\tau_k + l} \\ &= S\frac{1 - \alpha^{\tau_{k+1} - \tau_k - 1}}{1 - \alpha} - \sum_{l=0}^{\tau_{k+1} - \tau_k - 1} D_{\tau_k + l}\frac{\alpha^l - \alpha^{\tau_{k+1} - \tau_k - 1}}{1 - \alpha}. \end{aligned}$$

Note that the first hand side is defined only when $\tau_{k+1} - \tau_k - 2 \geq 0$, and must be interpreted as 0, when $\tau_{k+1} - \tau_k - 1 = 0$. However the right hand side is defined if

$\tau_{k+1} - \tau_k - 1 \geq 0$, and vanishes when $\tau_{k+1} - \tau_k - 1 = 0$. Therefore, formula (9.2.40) is valid for $\tau_{k+1} - \tau_k - 1 \geq 0$. From Lemma 9.5 we have

$$E[y_{\tau_k+1} + \cdots + y_{\tau_k+\tau_{k+1}-\tau_k-1}\alpha^{\tau_{k+1}-\tau_k-2}|\mathcal{F}^{\tau_k}] = E\left[S\frac{1-\alpha^\theta}{1-\alpha} - \sum_{l=0}^{\theta} D_{1+l}\frac{\alpha^l - \alpha^\theta}{1-\alpha}\right], \tag{9.2.41}$$

where $\theta = \tau_2 - 2$. Note that for $k \geq 1$,

$$y_{\tau_{k+1}} = (S - (D_{\tau_k} + \cdots + D_{\tau_k+\tau_{k+1}-\tau_k-1}))^+,$$

and

$$\alpha^{\tau_{k+1}-1}y_{\tau_{k+1}} = \alpha^{\tau_k-1}\alpha^{\tau_{k+1}-\tau_k-1}\alpha(S - (D_{\tau_k} + \cdots + D_{\tau_k+\tau_{k+1}-\tau_k-1}))^+.$$

Therefore we can assert that

$$E[\alpha^{\tau_{k+1}-1}y_{\tau_{k+1}}|\mathcal{F}^{\tau_k}] = \alpha^{\tau_k-1}\alpha E\alpha^\theta(S - (D_1 + \cdots + D_{1+\theta}))^+. \tag{9.2.42}$$

Similarly, for $k \geq 1$

$$\alpha^{\tau_{k+1}-2}LS_{\tau_{k+1}-1} = \alpha^{\tau_k-1}\alpha^{\tau_{k+1}-\tau_k-1}(S - (D_{\tau_k} + \cdots + D_{\tau_k+\tau_{k+1}-\tau_k-1}))^-.$$

Therefore

$$E[\alpha^{\tau_{k+1}-2}LS_{\tau_{k+1}-1}|\mathcal{F}^{\tau_k}] = \alpha^{\tau_k-1}E\alpha^\theta(S - (D_1 + \cdots + D_{1+\theta}))^-. \tag{9.2.43}$$

We can then assert that

$$I = (h-c)x + K + cS + p\sum_{k=1}^{\infty} E\alpha^{\tau_k-1}E\alpha^\theta(S - (D_1 + \cdots + D_{1+\theta}))^-;$$

$$II = \sum_{k=1}^{\infty} E\alpha^{\tau_k-1}E\alpha^\theta[\alpha(K + cS) + \alpha(h-c)(S - (D_1 + \cdots + D_{1+\theta}))^+];$$

$$III = h\alpha\sum_{k=1}^{\infty} E\alpha^{\tau_k-1}E\left[S\frac{1-\alpha^\theta}{1-\alpha} - \sum_{l=0}^{\theta} D_{1+l}\frac{\alpha^l - \alpha^\theta}{1-\alpha}\right].$$

Using next $\alpha^{\tau_{k+1}-1} = \alpha^{\tau_k-1}\alpha^{\tau_{k+1}-\tau_k-1}\alpha$, we get

$$E\alpha^{\tau_{k+1}-1} = E\alpha^{\tau_k-1}E\alpha\alpha^\theta,$$

and since $E\alpha^{\tau_1-1} = 1$, we obtain

$$E\alpha^{\tau_k-1} = (E\alpha\alpha^\theta)^{k-1}, \ k \geq 1. \tag{9.2.44}$$

Collecting results, we can assert that

$$\begin{aligned} J_x(V_{s,S}) &= (h-c)x + K + cS + \\ &+ \frac{1}{1 - E\alpha\alpha^\theta}E\alpha^\theta[\alpha(K + cS) + \alpha(h-c)(S - (D_1 + \cdots + D_{1+\theta}))^+] \\ &+ \frac{1}{1 - E\alpha\alpha^\theta}E\left[\alpha^\theta p(S - (D_1 + \cdots + D_{1+\theta}))^- \right.\\ &\left. + h\alpha\left(S\frac{1-\alpha^\theta}{1-\alpha} - \sum_{l=0}^{\theta} D_{1+l}\frac{\alpha^l - \alpha^\theta}{1-\alpha}\right)\right]. \end{aligned} \tag{9.2.45}$$

Lemma 9.6. *We have*

$$E\alpha\alpha^{\theta} = 1-(1-\alpha)\Gamma(S-s) \tag{9.2.46}$$

$$\begin{aligned} E\alpha^{\theta}\Phi(D_1+\cdots+D_{1+\theta}) &= \\ &-\frac{1}{\alpha}\Gamma(S-s)(\Phi(S-s)-\alpha E\Phi(S-s+D))+\frac{\Phi(0)}{\alpha} \\ &+\frac{1}{\alpha}\int_0^{S-s}\Gamma(x)(\Phi'(x)-\alpha E\Phi'(x+D)). \end{aligned} \tag{9.2.47}$$

PROOF. We can write

$$\begin{aligned} E\alpha^{\theta} &= \sum_{n=0}^{\infty}\alpha^n E\mathbb{1}_{\theta=n} \\ &= E\mathbb{1}_{D_1\geq S-s}+\sum_{n=1}^{\infty}\alpha^n E\mathbb{1}_{D_1+\cdots+D_{1+n}\geq S-s}\mathbb{1}_{D_1+\cdots+D_n<S-s} \\ &= 1-F(S-s)+\sum_{n=1}^{\infty}\alpha^n(F^{(n)}(S-s)-F^{(n+1)}(S-s)) \\ &= \sum_{n=0}^{\infty}\alpha^n(F^{(n)}(S-s)-F^{(n+1)}(S-s)) \\ &= \frac{1}{\alpha}-\left(\frac{1}{\alpha}-1\right)\sum_{n=0}^{\infty}\alpha^n F^{(n)}(S-s), \end{aligned}$$

and the property (9.2.46) follows. Next, we write

$$\begin{aligned} &E\alpha^{\theta}\Phi(D_1+\cdots+D_{1+\theta}) \\ &\quad=\sum_{n=0}^{\infty}\alpha^n E\mathbb{1}_{\theta=n}\Phi(D_1+\cdots+D_n)E\Phi(D_1)\mathbb{1}_{D_1\geq S-s} \\ &+\sum_{n=1}^{\infty}\alpha^n E\Phi(D_1+\cdots+D_n)\mathbb{1}_{D_1+\cdots+D_{1+n}\geq S-s}\mathbb{1}_{D_1+\cdots+D_n<S-s} \\ &\quad=E\Phi(D_1)\mathbb{1}_{D_1\geq S-s} \\ &+\sum_{n=1}^{\infty}\alpha^n E\Phi(D_1+\cdots+D_n)(\mathbb{1}_{D_1+\cdots+D_n<S-s}-\mathbb{1}_{D_1+\cdots+D_{1+n}<S-s}) \\ &\quad=E\Phi(D)\mathbb{1}_{D\geq S-s} \\ &+\sum_{n=1}^{\infty}\alpha^n\int_0^{S-s}f^{*(n)}(x)E\Phi(x+D)dx-\sum_{n=1}^{\infty}\alpha^n\int_0^{S-s}f^{*(n+1)}(x)\Phi(x)dx \\ &\quad=E\Phi(D)\mathbb{1}_{D\geq S-s} \\ &+\int_0^{S-s}\Gamma'(x)E\Phi(x+D)dx-\frac{1}{\alpha}\int_0^{S-s}(\Gamma'(x)-\alpha f(x))\Phi(x)dx \\ &\quad=E\Phi(D)-\frac{1}{\alpha}\int_0^{S-s}\Gamma'(x)(\Phi(x)-\alpha E\Phi(x+D))dx, \end{aligned}$$

and by integrating by parts we obtain (9.2.47). ∎

We need now to compute

$$\begin{aligned} E\sum_{l=0}^{\theta}\alpha^l D_{1+l} &= E\sum_{l=0}^{\infty}\alpha^l D_{1+l}1\!\!1_{\theta\geq l} \\ &= \bar{D} + E\sum_{l=1}^{\infty}\alpha^l D_{1+l}1\!\!1_{\theta\geq l} \\ &= \bar{D} + E\sum_{l=1}^{\infty}\alpha^l D_{1+l}1\!\!1_{D_1+\cdots D_{1+l}<S-s} \\ &= \bar{D} + \bar{D}\sum_{l=1}^{\infty}\alpha^l F^{(l)}(S-s). \end{aligned}$$

Therefore

$$E\sum_{l=0}^{\theta}\alpha^l D_{1+l} = \bar{D}\Gamma(S-s). \tag{9.2.48}$$

We can now collect results to compute

$$\begin{aligned} J_x(V_{s,S}) &= (h-c)x + K + cS \\ &+\frac{1}{(1-\alpha)\Gamma(S-s)}E\alpha^\theta\Phi(D_1+\cdots+D_{1+\theta}) \\ &+\frac{1}{(1-\alpha)\Gamma(S-s)}\Bigg[(K-\frac{S}{\alpha}(p+(h-c)\alpha))(1-(1-\alpha)\Gamma(S-s)) \\ &+\frac{h\alpha\Gamma(S-s)}{1-\alpha}(S(1-\alpha)-\bar{D})\Bigg], \end{aligned} \tag{9.2.49}$$

with

$$\Phi(x) = (p+\alpha(h-c))(S-x)^+ + \left(p+\frac{h\alpha}{1-\alpha}\right)x.$$

We have

$$\Phi'(x) = p + \frac{h\alpha}{1-\alpha} - (p+\alpha(h-c))1\!\!1_{S>x},$$

hence

$$\begin{aligned} \Phi(S-s) - \alpha E\Phi(S-s+D) &= (p+\alpha(h-c))(s-\alpha E(s-D)^+) \\ &+((1-\alpha)p+\alpha h)(S-s) - \alpha\bar{D}\left(p+\frac{h\alpha}{1-\alpha}\right), \end{aligned}$$

$$\Phi'(x) - \alpha E\Phi'(x+D) = (1-\alpha)p + h\alpha - (p+\alpha(h-c))(1-\alpha F(S-x))1\!\!1_{S>x}.$$

We can then apply formula (9.2.47) to obtain, after rearrangements

$$\begin{aligned} E\alpha^\theta\Phi(D_1+\cdots+D_{1+\theta}) &= \frac{1}{\alpha}S(p+\alpha(h-c)) \\ &-\frac{1}{\alpha}\Gamma(S-s)\Bigg[S((1-\alpha)p+h\alpha)+s\alpha(p-c) \\ &-\alpha(p+\alpha(h-c))E(s-D)^+ - \alpha\bar{D}\left(p+\frac{h\alpha}{1-\alpha}\right)\Bigg] \\ &+\int_0^{S-s}\Gamma(x)[c-p+(p+\alpha(h-c))F(S-x)]dx. \end{aligned}$$

Using this formula in (9.2.49) we deduce

$$J(V_{s,S}) = (h-c)x + cS + \frac{K}{(1-\alpha)\Gamma(S-s)}$$
$$-\frac{1}{1-\alpha}[cS(1-\alpha) + s(p-c) - (p+\alpha(h-c))E(s-D)^+ - p\bar{D}]$$
$$+\frac{1}{(1-\alpha)\Gamma(S-s)}\int_0^{S-s}\Gamma(x)[c-p+(p+\alpha(h-c))F(S-x)]dx$$

and equation (9.2.35) follows easily.

We now consider the case

II- $x > s$**:**

Now $\tau_1 \geq 2$. We have the definition

$$\tau_1 = 2 + \inf\{n \geq 0 | D_1 + \cdots + D_{1+n} \geq x - s\}.$$

Moreover

$$y_{\tau_1} = (x - (D_1 + \cdots + D_{\tau_1 - 1}))^+ \qquad LS_{\tau_1 - 1} = (x - (D_1 + \cdots + D_{\tau_1 - 1}))^-,$$

and $v_{\tau_1} = S - y_{\tau_1}$. Moreover if $\tau_1 \geq 3$, we have

$$\begin{aligned} y_2 &= x - D_1; \\ \cdots & \\ y_{\tau_{1}-1} &= x - (D_1 + \cdots + D_{\tau_1 - 2}). \end{aligned}$$

By construction $D_1 + \cdots + D_{\tau_1 - 1} \geq x - s$. Therefore $y_{\tau_1} \leq s$, and τ_1 is the first ordering point. It is a $\mathcal{F}^{n-1}$ stopping time. From τ_1 on the sequence $\tau_k, k \geq 2$ is defined as in (9.2.37). The quantities $y_{\tau_k}, LS_{\tau_k - 1}, v_{\tau_k}$ are defined as in the first part, with of course $\tau_1 \neq 1$ and given by the new definition. We can then write

$$J_x(V_{s,S}) = J_x^1(V_{s,S}) + J_x^2(V_{s,S}), \tag{9.2.50}$$

with

$$\begin{aligned} J_x^1(V_{s,S}) &= hx + hE(\alpha y_2 + \cdots + \alpha^{\tau_1 - 2}) \\ &+ E\alpha^{\tau_1 - 1}(K + cS + (h-c)y_{\tau_1}) + pE\alpha^{\tau_1 - 2} LS_{\tau_1 - 1}, \end{aligned} \tag{9.2.51}$$

$$\begin{aligned} J_x^2(V_{s,S}) &= p\sum_{k=2}^{\infty} E\alpha^{\tau_k - 2} LS_{\tau_k - 1} + \sum_{k=2}^{\infty} E\alpha^{\tau_k - 1}(K + cS + (h-c)y_{\tau_k}) \\ &+ h\sum_{k=1}^{\infty} E\alpha^{\tau_k - 1}(\alpha y_{\tau_k + 1} + \cdots + y_{\tau_k + \tau_{k+1} - \tau_k - 1}\alpha^{\tau_{k+1} - \tau_k - 1}). \end{aligned} \tag{9.2.52}$$

We begin with $J_x^2(V_{s,S})$, which is closely related to the computations of the first part. We have

$$J_x^2(V_{s,S}) = [\Gamma(S-s)(g(s) - (K+cS)(1-\alpha)) + K + H_s(S)]\sum_{k=1}^{\infty} E\alpha^{\tau_k - 1},$$

and

$$\sum_{k=1}^{\infty} E\alpha^{\tau_k - 1} = \frac{E\alpha^{\tau_1 - 1}}{(1-\alpha)\Gamma(S-s)}.$$

Therefore

$$J_x^2(V_{s,S}) = E\alpha^{\tau_1-1}\left[\frac{g(s)}{1-\alpha} - (K+cS) + \frac{K+H_s(S)}{(1-\alpha)\Gamma(S-s)}\right]. \tag{9.2.53}$$

Now $\tau_1 - 2$ and θ have the same distribution with S replaced with x. So we can write the formulas

$$E\alpha^{\tau_1-2} = \frac{1-(1-\alpha)\Gamma(x-s)}{\alpha}; \tag{9.2.54}$$

$$\begin{aligned} &E\alpha^{\tau_1-2}\Phi(D_1+\cdots+D_{1+\tau_1-2}) = \\ &\quad -\frac{1}{\alpha}\Gamma(x-s)(\Phi(x-s)-\alpha E\Phi(x-s+D)) \\ &\quad +\frac{\Phi(0)}{\alpha} + +\frac{1}{\alpha}\int_0^{x-s}\Gamma(x)(\Phi'(x)-\alpha E\Phi'(x+D)). \end{aligned} \tag{9.2.55}$$

It is then possible to compute $J_x^1(V_{s,S})$, with calculations similar to those of the first part and to obtain the formula (9.2.36). This concludes the proof of Theorem 9.7 □

It is clear from formulas (9.2.35) and (9.2.36) that one can optimize S independently of x. It is sufficient to minimize the function

$$Z_s(x) = \frac{K+H_s(x)}{\Gamma(x-s)}, \tag{9.2.56}$$

for $x \geq s$. It is easy to check that there is a minimum, and we can always take the smallest minimum. Indeed, one has $Z_s(s) = K$ and $Z_s(x) \to +\infty$ as $x \to +\infty$, since $\Gamma(x-s) \to \dfrac{1}{1-\alpha}$, as $x \to +\infty$. So in fact, there is only one remaining choice, the value of s. We can then define the function $u_s(x)$ by the formulas

$$\begin{aligned} u_s(x) &= (h-c)x + H_s(x) + \frac{g(s)}{1-\alpha} \\ &\quad + \frac{(1-(1-\alpha)\Gamma(x-s))}{1-\alpha}\inf_{\xi\geq s}\frac{K+H_s(\xi)}{\Gamma(\xi-s)}, \quad \forall x > s \end{aligned} \tag{9.2.57}$$

$$\begin{aligned} u_s(x) &= (h-c)x + \frac{g(s)}{1-\alpha} \\ &\quad + \frac{1}{1-\alpha}\inf_{\xi\geq s}\frac{K+H_s(\xi)}{\Gamma(\xi-s)}, \quad \forall x \leq s \end{aligned} \tag{9.2.58}$$

There is a discontinuity for $x = s$, unless

$$\inf_{\xi\geq s}\frac{K+H_s(\xi)}{\Gamma(\xi-s)} = 0, \tag{9.2.59}$$

which is equivalent to the condition $K + \inf_{\xi\geq s} H_s(\xi) = 0$. We recover the pair s, S obtained by the analytic theory of Impulse Control, Theorem 9.4. So the choice of s is dictated by a continuity condition of the cost function, and not by an optimization (unlike the choice of S). However, we have shown in Theorem 9.4, that the function defined by this s, S control policy is the value function and is solution of the Bellman equation.

9.2.5. INVARIANT MEASURE. The inventory, controlled by an s, S policy, remains in the interval $X = [0, S]$, if the initial state $y_1 = x \in X$. This state space being compact, the process is ergodic. The situation is very similar to that of section 3.6.1 of Chapter 3. The example developed in that section was in fact the case of $s = 0$. The controlled Markov chain is defined by the transition probability

$$\pi_{s,S}(x, d\eta) = \begin{cases} \bar{F}(x)\delta(\eta) + f(x-\eta)\mathbb{1}_{\eta<x}d\eta, & \text{if } x > s \\ \bar{F}(S)\delta(\eta) + f(S-\eta)\mathbb{1}_{\eta<S}d\eta, & \text{if } x \leq s \end{cases} \tag{9.2.60}$$

To simplify notation we shall omit to mention explicitly the indices s, S. The operator $\Phi = \Phi_{s,S}$ associated to the Markov chain is (see Chapter 3)

$$\Phi\varphi(x) = \mathbb{1}_{x>s}\left[\varphi(0)\bar{F}(x) + \int_0^x \varphi(\eta)f(x-\eta)d\eta\right] \tag{9.2.61}$$
$$+\mathbb{1}_{x\leq s}\left[\varphi(0)\bar{F}(S) + \int_0^S \varphi(\eta)f(S-\eta)d\eta\right].$$

Even if φ is continuous the image $\Phi\varphi(.)$ is discontinuous in s. Assuming for instance $\bar{F}(S) > 0$, suffices to imply the ergodicity, see Chapter 3. An invariant measure $m(dx)$ on X, must satisfy the equation

$$\int_0^S \varphi(\eta)m(d\eta) = \int_0^S\int_0^S m(dx)\pi(x;d\eta)\varphi(\eta), \forall\varphi(.) \text{ bounded.} \tag{9.2.62}$$

From ergodic theory, we know that there exists one and only one measure solution of (9.2.62). We postulate the form

$$m(dx) = A\delta(x) + B(x)dx, \tag{9.2.63}$$

where $\delta(x)$ is the Dirac measure at 0. Since $m(dx)$ is a probability, we must have the relation

$$A + \int_0^S B(x)dx = 1. \tag{9.2.64}$$

We introduce the integral equation

$$z(\eta) = \frac{f(S-\eta)}{f(0)} + \int_s^S \mathbb{1}_{x>\eta}f(x-\eta)z(x)dx, \tag{9.2.65}$$

and we look for solutions in the space $B(0, S)$ of measurable bounded functions on $(0, S)$. We have the

Theorem 9.8. *We assume $\bar{F}(S) > 0$. Then equation (9.2.65) has a unique solution in $B(0, S)$. Define next*

$$B(S) = \frac{F(S)}{\int_0^S \bar{F}(x)z(x)dx + \int_0^s F(x)z(x)dx + \int_s^S F(S)z(x)dx}, \tag{9.2.66}$$

and $B(x) = B(S)z(x), x \in (0, S)$. Then the measure $m(dx)$ defined by (9.2.63) is the unique invariant measure.

PROOF. The map

$$z(.) \Longrightarrow T(z)(.),$$

defined by

$$T(z)(\eta) = \frac{f(S-\eta)}{f(0)} + \int_s^S \mathbb{1}_{x>\eta}f(x-\eta)z(x)dx,$$

satisfies

$$||T(z_1) - T(z_2)|| \leq F(S)||z_1 - z_2||,$$

and from the assumption it is a contraction. Hence it has a fixed point. One checks easily that the function $B(x)$ satisfies

$$B(\eta) + \int_s^S B(x)[f(S-\eta) - f(x-\eta)\mathbb{1}_{x>\eta}]dx = f(S-\eta), \tag{9.2.67}$$

and we have

$$A = \bar{F}(S) + \int_s^S B(x)(F(S) - F(x))dx. \tag{9.2.68}$$

After rearrangements we have also

$$A = \frac{\int_s^S \bar{F}(x)B(x)dx + \bar{F}(S)\int_0^s B(x)dx}{F(S)};$$

$$B(\eta) = f(S-\eta)\left(A + \int_0^s B(x)dx\right) + \int_s^S f(x-\eta)B(x)\mathbb{1}_{x>\eta}dx,$$

and for any function $\varphi(.)$ we can then check the relation (9.2.62). Hence $m(dx)$ is an invariant measure. It is the only one, by ergodicity property. □

Consider now the objective function (9.2.7), when we apply a control $V = V_{s,S}$ defined by an s, S policy. The inventory y_n becomes ergodic so we have

$$(1-\alpha)\int_0^S J_x(V_{s,S})m_{s,S}(dx) = \int_0^S \big[hx + (K + c(S-x))\mathbb{1}_{x\leq s}$$
$$+p(E(S-D)^-\mathbb{1}_{x\leq s} + E(x-D)^-\mathbb{1}_{x>s})\big]m_{s,S}(dx).$$

From the property (9.2.62) of the invariant measure, this expression is also equal to

$$(1-\alpha)\int_0^S J_x(V_{s,S})m_{s,S}(dx) = p\bar{D} + h\int_0^S xm_{s,S}(dx)$$
$$+\int_0^s (K + (c-p)(S-x))m_{s,S}(dx). \tag{9.2.69}$$

We omit in the following the indices s, S to simplify notation. Using the definition of the function $g(x)$, see (9.2.13), we have also

$$(1-\alpha)\int_0^S J_x(V)m(dx) = (h-c)(1-\alpha)\int_0^S xm(dx) + K\int_0^s m(dx)$$
$$+\int_0^S (g(x)\mathbb{1}_{x>s} + g(S)\mathbb{1}_{x\leq s})m(dx).$$

Using now the function $H(x)$ which satisfies

$$H(x) = (g(x) - g(s))\mathbb{1}_{x>s} + \alpha EH((x-D)^+),$$

we obtain also

(9.2.70)
$$\int_0^S J_x(V)m(dx) = \int_0^S ((h-c)x + H(x))m(dx) + \frac{g(s)}{1-\alpha} + \frac{K + H(S)}{1-\alpha}\int_0^s m(dx).$$

Exercise 9.4. Show directly formula (9.2.70) from the formulas giving $J_x(V)$, formulas (9.2.35), (9.2.36). One needs to prove that the function $\Gamma(x)$ satisfies the relation

$$\Gamma(S-s)\int_0^s m(dx) + (1-\alpha)\int_s^S \Gamma(x-s)m(dx) = 1. \tag{9.2.71}$$

Indeed, from the equation of Γ, one can state

$$(\Gamma(x-s)-1)1\!\!1_{x>s} = \alpha 1\!\!1_{x>s} E\Gamma(x-s-D)1\!\!1_{x-s-D>0},$$

and also

$$\Gamma(S-s)-1 = \alpha E\Gamma(S-s-D)1\!\!1_{S-s-D>0}.$$

Therefore

$$(\Gamma(x-s)-1)1\!\!1_{x>s} + 1\!\!1_{x\le s}(\Gamma(S-s)-1) = \alpha[1\!\!1_{x>s}E\Gamma(x-s-D)1\!\!1_{x-s-D>0} + 1\!\!1_{x\le s}E\Gamma(S-s-D)1\!\!1_{S-s-D>0}].$$

Integrating both sides with respect to the invariant measure yields

$$\int_s^S \Gamma(x-s)m(dx) + \Gamma(S-s)\int_0^s m(dx) - 1 = \alpha\int_s^S \Gamma(x-s)m(dx),$$

where we have used the fundamental property of the invariant measure(9.2.62) applied to the test function $\varphi(x) = \Gamma(x-s)1\!\!1_{x-s>0}$. This is the result (9.2.71).

9.2.6. PARTICULAR CASE. Consider the case $f(x) = \beta \exp -\beta x$. We can then obtain analytic formulas. We first have

$$\Gamma(x) = \frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\exp -\beta(1-\alpha)x, \tag{9.2.72}$$

then

$$\mu(x) = c(1-\alpha) + \alpha h - (p-\alpha(c-h))\exp -\beta x; \tag{9.2.73}$$

$$g(x) = p\bar{D} - \frac{p-\alpha(c-h)}{\beta} + (c(1-\alpha)+\alpha h)x + \frac{p-\alpha(c-h)}{\beta}\exp -\beta x. \tag{9.2.74}$$

The value $\bar{s}$ such that $\mu(\bar{s}) = 0$ is given by

$$\exp -\beta\bar{s} = \frac{c(1-\alpha)+\alpha h}{p-\alpha(c-h)}. \tag{9.2.75}$$

Next we have, for $x \geq s$

$$H(x) = \frac{c(1-\alpha)+\alpha h}{1-\alpha}(x-s) - \frac{1-\exp -\beta(1-\alpha)(x-s)}{\beta(1-\alpha)} \cdot \left[\frac{\alpha}{1-\alpha}(c(1-\alpha)+\alpha h) + (p-\alpha(c-h))\exp -\beta s\right]. \tag{9.2.76}$$

For $s \leq \bar{s}$, The number S is uniquely defined by the equation $H'(S) = 0$, namely

$$\exp -\beta(1-\alpha)(S-s) = \frac{c(1-\alpha)+\alpha h}{\alpha(c(1-\alpha)+\alpha h) + (1-\alpha)(p-\alpha(c-h))\exp -\beta s}. \tag{9.2.77}$$

We deduce

$$H(S) = \frac{c(1-\alpha)+\alpha h}{1-\alpha}(S-s) - \frac{(p-\alpha(c-h))\exp -\beta s - (c(1-\alpha)+\alpha h)}{\beta(1-\alpha)}, \tag{9.2.78}$$

and s is defined by the equation $H_s(S(s)) = -K$, assuming that $H_0(S(0)) < -K$. The number $S_0 = S(0)$ is defined by

$$\exp -\beta(1-\alpha)S_0 = \frac{c(1-\alpha)+\alpha h}{p(1-\alpha)+\alpha h}, \tag{9.2.79}$$

and the condition amounts to

$$\frac{c(1-\alpha)+\alpha h}{1-\alpha}S_0 - \frac{p-c}{\beta(1-\alpha)} < -K. \tag{9.2.80}$$

Let us now turn to the invariant measure. The solution of (9.2.65) is

$$z(\eta) = \begin{cases} 1, & \text{if } \eta > s \\ \exp -\beta(s-\eta), & \text{if } \eta \leq s \end{cases} \tag{9.2.81}$$

Then $B(S) = \dfrac{\beta}{1+\beta(S-s)}$ and thus

$$B(\eta) = \begin{cases} \dfrac{\beta}{1+\beta(S-s)}, & \text{if } \eta > s \\ \dfrac{\beta\exp -\beta(s-\eta)}{1+\beta(S-s)}, & \text{if } \eta \leq s \end{cases} \qquad A = \frac{\exp -\beta s}{1+\beta(S-s)}. \tag{9.2.82}$$

9.3. INVENTORY CONTROL WITH FIXED COST AND BACKLOG

9.3.1. BELLMAN EQUATION. We consider the situation of section 5.2, in which we add a fixed cost. The Bellman functional equation is given by

$$u(x) = \inf_{v\geq 0}[l(x,v) + \alpha Eu(x+v-D)], \tag{9.3.1}$$

with

$$l(x,v) = K\mathbb{1}_{v>0} + cv + hx^+ + px^-. \tag{9.3.2}$$

We write also (9.3.1) as

$$u(x) = hx^+ + px^- + \inf_{v\geq 0}[K\mathbb{1}_{v>0} + cv + \alpha Eu(x+v-D)]. \tag{9.3.3}$$

We define the ceiling function by the linear equation

$$w_0(x) = hx^+ + px^- + \alpha Ew_0(x-D). \tag{9.3.4}$$

Recalling

$$\Phi^v\varphi(x) = E\varphi(x+v-D), \tag{9.3.5}$$

then

$$w_0(x) = \sum_{n=1}^{\infty}\alpha^{n-1}(\Phi^0)^{n-1}(hx^+ + px^-)(x), \tag{9.3.6}$$

and the series is well defined with

$$w_0(x) \leq \max(h,p)\left[\frac{|x|}{1-\alpha} + \frac{\bar{D}}{(1-\alpha)^2}\right]. \tag{9.3.7}$$

Assumptions (4.3.1), (4.3.2), (4.3.6) are satisfied. So results of Theorems 4.3, 4.4, 4.5 are applicable. We can then state

Theorem 9.9. *Under the assumptions* (9.3.1), (9.3.5) *the set of solutions of the functional equation* (9.3.3)*, in the interval* $0 \leq u \leq w_0$ *is not empty and has a minimum and a maximum solution* $\underline{u}$ *and* $\overline{u}$*. The minimum solution is l.s.c. It coincides with the value function. There exists an optimal control policy, obtained through a feedback* $\hat{v}(x)$*.*

We can check an estimate for the optimal feedback as in the case without fixed cost.

Proposition 9.4. *The optimal feedback satisfies*

$$\hat{v}(x) \leq x^- + \frac{\alpha\bar{D}}{c(1-\alpha)^2}[h+(p+c)(1-\alpha)] + \frac{\alpha K}{1-\alpha}. \tag{9.3.8}$$

PROOF. We begin by getting an estimate from below of the value function. It is the same as in the case without fixed cost, with the same proof,

$$u(x) \geq \frac{hx^+}{1-\alpha} + px^- - \frac{h\alpha\bar{D}}{(1-\alpha)^2}.$$

The next step is to majorize $u(x)$ by a convenient function. We do not use $w_0(x)$ but the function $w(x)$ solution of

$$w(x) = K\mathbb{1}_{x<0} + (c+p)x^- + hx^+ + \alpha E w(x^+ - D).$$

We have that $u(x) \leq w(x)$. The following estimate holds

$$w(x) \leq (c+p)x^- + \frac{hx^+}{1-\alpha} + \frac{K + \alpha(c+p)\bar{D}}{1-\alpha}.$$

Now, going back to the functional equation (9.3.3), we minorize the quantity within brackets on the right hand side, by using

$$u(x) \geq \frac{hx^+}{1-\alpha} - \frac{h\alpha\bar{D}}{(1-\alpha)^2}.$$

We see that it is minorized by

$$K + cv + \frac{hx^+}{1-\alpha} + px^- - \frac{h\alpha\bar{D}}{(1-\alpha)^2}.$$

Therefore, looking for the infimum, we can restrict the controls v to the set

$$K + cv + \frac{hx^+}{1-\alpha} + px^- - \frac{h\alpha\bar{D}}{(1-\alpha)^2} \leq w(x).$$

Using then the estimate on $w(x)$, we deduce that we can restrict the set of v to be bounded by the right hand side of (9.3.8). The optimal feedback must satisfy this bound, hence the result (9.3.8).

We can then prove the uniqueness of the solution of (9.3.3) in the space B_1, where

$$u \in B_1 \Longleftrightarrow \frac{|u(x)|}{1+|x|} \leq C.$$

□

Theorem 9.10. *Under the assumptions* (9.3.1), (9.3.5)*, the solution of* (9.3.3) *in the space* B_1 *is unique. The function* u *is l.s.c*

PROOF. As usual, we are going to check that the minimum and the maximum solution coincide. Define the optimal control associated with the optimal feedback. If we denote the optimal control by $\hat{V} = \{\hat{v}_1, \cdots, \hat{v}_n, \cdots\}$ and the optimal trajectory by $\{\hat{y}_1, \cdots, \hat{y}_n, \cdots\}$ we must prove that $\hat{V}$ belongs to $\mathcal{V}$, which means that

$$\alpha^{j-1} E|\hat{y}_j| \to 0,$$

as $j \to \infty$. However, from the estimate (9.3.8) and setting

$$C = \frac{\alpha \bar{D}}{c(1-\alpha)^2}[h + (p+c)(1-\alpha)] + \frac{\alpha K}{1-\alpha},$$

we can state that

$$\hat{y}_{n+1} \leq \hat{y}_n^+ + C,$$

hence

$$\hat{y}_{n+1} \leq x^+ + nC.$$

On the other hand

$$\hat{y}_{n+1} \geq x - D_1 - \cdots - D_n.$$

Therefore

$$x - n\bar{D} \leq E\hat{y}_{n+1} \leq x^+ + nC,$$

and the condition is satisfied, hence the result. □

9.3.2. s, S **Policy.** Our objective is now to obtain the optimal feedback. We want to show that it is derived from an s, S policy.

First of all we introduce the operators

$$T(u)(x) = hx^+ + px^- + \alpha Eu(x-D), \tag{9.3.9}$$

and

$$M(u)(x) = \inf_{\eta \geq x}[K1\!\!1_{\eta > x} + (c-h)(\eta^+ - x^+) - (p+c)(\eta^- - x^-) + u(\eta)]. \tag{9.3.10}$$

This operator is a little bit more involved than the one introduced for the case without shortage (see (9.2.10). The variable cost from going to an inventory x to an inventory $\eta \geq x$ is positive, if the assumption $c > h$ holds, but is not a function of the difference $\eta - x$, because the inventories do not have the same meaning when they are positive or negative.

The functional equation (9.3.3) can be written as

$$u(x) = \min[T(u)(x), M(T(u))(x)],$$

and we see as in (9.2.11) that this is equivalent to

$$u(x) = \min[T(u)(x), M(u)(x)].$$

Define next

$$g(x) = cx(1-\alpha) + \alpha c\bar{D} + \alpha hE(x-D)^+ + \alpha pE(x-D)^-, \tag{9.3.11}$$

and

$$g_s(x) = (g(x) - g(s))1\!\!1_{x>s}. \tag{9.3.12}$$

We then define the function $H_s(x)$ by solving

$$H_s(x) = g_s(x) + \alpha EH_s(x-D). \tag{9.3.13}$$

We know that the solution is given by

$$H_s(x) = g_s(x) + \alpha g_s \star f + \sum_{n=2}^{\infty} \alpha^n g_s \star f^{\star(n)}(x), \tag{9.3.14}$$

and using

$$F^{(n)}(x) = \int_0^x f^{\star(n)}(\xi)\, d\xi,$$

and

$$\Gamma(x) = \sum_{n=0}^{\infty} \alpha^n F^{(n)}(x),$$

then we can easily verify the formula

$$H_s(x) = \int_s^x \Gamma(x-\xi)\mu(\xi)d\xi, \tag{9.3.15}$$

with $\mu = g'$. The function $\Gamma(x)$ is the solution of the renewal equation (9.2.19).

Lemma 9.7. *Assume*

$$h < c < p\frac{\alpha}{1-\alpha}, \tag{9.3.16}$$

then the function $H_s(x)$ attains its minimum for $x \geq s$. Taking the smallest minimum, one defines a function $S(s)$. If $\bar{s}$ is the unique value such that $\mu(s) = 0, \bar{s} > 0$, then $S(s) = s, \forall s \geq \bar{s}$. The equation

$$H_s(S(s)) = \inf_{\eta \geq s} H_s(\eta) = -K, \tag{9.3.17}$$

has a unique solution $s < \bar{s}$.

Proof. It is similar to the proof of Lemma 9.4. We have

$$g'(x) = \mu(x) = c(1-\alpha) + \alpha h - \alpha(h+p)\bar{F}(x). \tag{9.3.18}$$

Thanks to the condition (9.3.16) we can define $\bar{s} > 0$ such that $\mu(\bar{s}) = 0$ and $\mu(x) < 0, \forall x < \bar{s}$ and $\mu(x) > 0, \forall x > \bar{s}$. Since Γ is an increasing function, the function $H_s(x)$ is increasing in x, whenever $s \geq \bar{s}$. Therefore we can assert

$$S(s) = s, \;\; \inf_{\eta \geq s} H_s(\eta) = 0, \text{ if } s \geq \bar{s}.$$

So we limit the set of possible s to satisfy $s < \bar{s}$. Clearly for $s < \bar{s} < x$

$$H_s(x) = H_{\bar{s}}(x) + \int_s^{\bar{s}} \Gamma(x-\xi)\mu(\xi)d\xi.$$

We have again

$$H_s(x) \geq \frac{1}{1-\alpha}(g(\bar{s}) - g(s)) + g(x) - g(s).$$

Therefore $H_s(x) \to \infty$ as $x \uparrow \infty$. Since $H_s^{'}(s+) = \mu(s) < 0$ the function $H_s(x)$ attains a negative minimum on (s, ∞). We call $S(s)$ the smallest minimum if there are many. We have

$$H_s'(S(s)) = 0.$$

We have next

$$\frac{d}{ds}H_s(S(s)) = \frac{\partial}{\partial s}H_s(x)|_{S(s)} = -\Gamma(S(s)-s)\mu(s) > 0.$$

Also

$$\begin{aligned} H_s(S(s)) &\leq H_s(\bar{s}) = \int_s^{\bar{s}} \Gamma(\bar{s}-\xi)\mu(\xi)d\xi \\ &\leq \frac{1}{\alpha}\int_s^{\bar{s}} \mu(\xi)d\xi = \frac{1}{\alpha}(g(\bar{s})-g(s)). \end{aligned}$$

Since $g(s) \to +\infty$, as $s \to -\infty$, we have $H_s(S(s)) \to -\infty$, as $s \to -\infty$.

Hence the function $\inf_{\eta\geq s} H_s(\eta)$ increases from $-\infty$ to 0, as s increases from $-\infty$ to $\bar{s}$. Therefore there exists a unique s for which it is equal to $-K$. We can find the pair s, S by solving the algebraic system

$$\begin{aligned} &\Gamma(S-s)\mu(s) + \int_s^S \Gamma(S-\xi)\mu'(\xi)d\xi = 0; \\ &\int_s^S \Gamma(S-\xi)\mu(\xi)d\xi = -K. \end{aligned} \tag{9.3.19}$$

We can finally state the □

Theorem 9.11. *Under the assumptions* (9.3.2), (9.3.5) *and* (9.3.16) *the functional equation* (9.3.3) *has a unique solution in* B_1*, which is continuous, tends to* $+\infty$ *as* $|x|$ *tends to* $+\infty$*. The optimal feedback is defined by an* s, S *policy, which is a solution of the system* (10.3.22).

PROOF. Having defined s, S by solving (10.3.22), which defines uniquely s, and then S, taking possibly the smallest value, we consider $H_s(x)$ given by (9.3.15). We will from now on drop the index s, and thus consider $H(x)$. We then define

$$u(x) = H(x) + hx^+ + px^- - cx + \frac{g(s)}{1-\alpha}. \tag{9.3.20}$$

Then $u(x)$ satisfies

$$u(x) = \begin{cases} hx^+ + px^- - cx + \dfrac{g(s)}{1-\alpha}, & x \leq s \\ hx^+ + px^- + \alpha E u(x-D), & x \geq s \end{cases} \tag{9.3.21}$$

This function is continuous and tend to $+\infty$ as $|x| \to +\infty$.

And also, from the choice of s, for $x \leq s$

$$\begin{aligned} u(x) &= K + \inf_{\eta\geq s}[u(\eta) + (c-h)(\eta^+ - x^+) - (c+p)(\eta^- - x^-)] \\ &= K + \inf_{\eta\geq x}[u(\eta) + (c-h)(\eta^+ - x^+) - (c+p)(\eta^- - x^-)]. \end{aligned}$$

Therefore the function u satisfies

$$u(x) = \begin{cases} M(u)(x), & \forall x \leq s \\ T(u)(x), & \forall x \geq s \end{cases}$$

So it remains to prove that

$$\begin{aligned} u(x) &\leq T(u)(x), \quad \forall x \leq s \\ u(x) &\leq M(u)(x), \quad \forall x \geq s \end{aligned}$$

In terms of H the first inequality amounts to $g_s(x) \geq 0, \forall x \leq s$. This is true since $\mu(\xi) < 0, \forall x \leq \xi \leq s < \bar{s}$. The second inequality in terms of H amounts to

$$H(x) \leq K + \inf_{\eta\geq x} H(\eta), \forall x \geq s. \tag{9.3.22}$$

To prove (9.3.22) we first notice that for $x \geq s$ we have $\inf_{\eta \geq x} H(\eta) \geq \inf_{\eta \geq s} H(\eta) = -K$. So the inequality is certainly true whenever $H(x) \leq 0$. Consider then the smallest number called x_0 for which H becomes positive. We have then

$$H(x_0) = 0,\ H(x) < 0 \forall x, s < x < x_0.$$

We can claim that $g_s(x_0) > 0$. This follows from the relation $g_s(x_0) = -\alpha EH(x_0 - D)$ and the properties of x_0 resulting from its definition.

Let $\xi > 0$. We shall consider the domain $x \geq x_0 - \xi$. We can write the inequality

$$H(x) \leq g_s(x) + \alpha E(H(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}), \forall x \geq x_0 - \xi,$$

which follows from the fact that

$$E(H(x-D)\mathbb{1}_{x-D \leq x_0 - \xi}) \leq 0.$$

Define next

$$M(x) = H(x+\xi) + K.$$

Clearly $M(x) \geq 0$. We can write thanks to this positivity

$$EM(x-D) \geq E(M(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}).$$

Now for $x \geq x_0 - \xi$ we have

$$H(x+\xi) - \alpha EH(x+\xi-D) = g_s(x+\xi),$$

hence

$$M(x) - \alpha EM(x-D) = g_s(x+\xi) + K(1-\alpha).$$

Therefore we have

$$M(x) \geq g_s(x+\xi) + \alpha E(M(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}), \forall x \geq x_0 - \xi.$$

We can claim that

$$g_s(x+\xi) \geq g_s(x), \forall x \geq x_0 - \xi.$$

Indeed, note that $g_s(x+\xi) \geq g_s(x_0) \geq 0$ since g is increasing on $[\bar{s}, \infty)$. So the inequality is obvious, whenever $x < s$. If $x \geq s$ the inequality amounts to $g(x+\xi) \geq g(x)$. This is true whenever $x \geq \bar{s}$, by the monotonicity property. So it remains the case $s < x < \bar{s}$. But in this case $g_s(x) < 0$ and the inequality is satisfied.

Let us define $Y(x) = H(x) - M(x)$. From the inequalities above we can state that

$$Y(x) \leq \alpha E(Y(x-D)\mathbb{1}_{x-D \geq x_0 - \xi}), \forall x \geq x_0 - \xi,$$

and $Y(x_0 - \xi) \leq -K$.

These two properties imply $Y(x) \leq 0$, $\forall x \geq x_0 - \xi$ Therefore we have obtained (9.3.22) $\forall x \geq x_0 - \xi$ and in particular for $x > x_0$ as desired. So u is solution of the functional equation (9.3.3) and thus is the value function. The feedback defined from the s, S policy is optimal. ☐

Remark. *Another proof of* (9.3.22)

We will give another proof of (9.3.22), using an idea originated by [**3**].

PROOF. We recall that

$$H(x) = g_s(x) + \alpha EH(x-D),$$

and we want to prove

$$H(x) \leq K + \inf_{\eta \geq x} H(\eta).$$

Define

$$B(x) = H(x) - \inf_{\eta \geq x} H(\eta).$$

So we want to prove

$$B(x) \leq K.$$

For $x \leq s$ we have

$$B(x) = -\inf_{\eta \geq x} H(\eta) = -\inf_{\eta \geq s} H(\eta) = K.$$

Consider the first point x_0 such that $x_0 > s$ and $H(x_0) = 0,\ H(x) \leq 0, \forall x \leq x_0$. For $s \leq x \leq x_0$ we have

$$B(x) \leq -\inf_{\eta \geq x} H(\eta) \leq -\inf_{\eta \geq s} H(\eta) = K.$$

It remains to consider the case $x > x_0$. Assume that there exists a point $x' > x_0$ such that $B(x') \geq K$. Then there will exist a point x' such that

$$\bar{s} < x_0 \leq x',\ B(x') = 0,\ B(x) < K,\ x < x'.$$

Define the point x_3 such that

$$H(x_3) = \inf_{\eta \geq x'} H(\eta).$$

The point x_3 is uniquely defined as the smallest infimum beyond x'. Moreover $x_3 \neq x'$ since

$$B(x') = H(x') - H(x_3) = K.$$

Next define x_2 such that

$$H(x_2) = \max_{\eta \leq x_3} H(\eta) = \max_{s \leq \eta \leq x_3} H(\eta).$$

We take x_2 the smallest maximum. Then we can assert that $x_2 \geq x'$. Indeed if $x_2 < x'$ then

$$\inf_{\eta \geq x_2} H(\eta) \leq \inf_{\eta \geq x'} H(\eta),$$

and

$$H(x_2) - \inf_{\eta \geq x_2} H(\eta) \geq H(x') - \inf_{\eta \geq x'} H(\eta) = K,$$

which will contradict the definition of x'. Hence $x' \leq x_2 \leq x_3$. Moreover $x_2 \neq x_3$ since

$$H(x_2) - H(x_3) \geq H(x') - H(x_3) = B(x') = K.$$

Let us check now that

$$H(x) - H(y) \leq H(x_2) - H(x_3),\ \forall x \leq y \leq x_3. \tag{9.3.23}$$

Indeed if $x < x'$ then

$$H(x) - H(y) \leq H(x) - \inf_{\eta \geq x} H(\eta) < K \leq H(x_2) - H(x_3).$$

Next if $x' \leq x \leq y \leq x_3$ then

$$H(x) - H(y) \leq H(x_2) - H(x_3).$$

Collecting results we note

$$s < \bar{s} < x_0 \leq x' \leq x_2 < x_3.$$

Now

$$\begin{aligned} H(x_2) &= g(x_2) - g(s) + \alpha E H(x_2 - D); \\ H(x_3) &= g(x_3) - g(s) + \alpha E H(x_3 - D), \end{aligned}$$

and we have

$$\begin{aligned} &g(x_2) - g(x_3) < 0; \\ &H(x_2 - D) - H(x_3 - D) < H(x_2) - H(x_3), \end{aligned}$$

from which, subtracting the equations we get $H(x_2) < H(x_3)$. This is a contradiction from the definition of x_2. □

We finally have the equivalent of Proposition 9.3

Proposition 9.5. *The function $S(s)$ is decreasing, hence $S(s) \geq \bar{s}$, if $s \leq \bar{s}$ and $S(s) \to S^*$, as $s \to -\infty$, with*

(9.3.24)
$$\Gamma(S^*) = \frac{p - (c-h)(1-\alpha)}{(1-\alpha)(h+p)}.$$

Proof. The first property is proven as in Proposition 9.3. The second property follows from the first equation (10.3.22), which defines $S(s)$. From the monotonicity we know that $S(s) \uparrow S^*$. Then

$$\Gamma(S - s)\mu(s) \uparrow \frac{c(1-\alpha) - \alpha p}{1-\alpha}.$$

We can go to the limit in the first equation (10.3.22) and obtain

$$\frac{c(1-\alpha) - \alpha p}{1-\alpha} + \alpha(h+p)\int_0^{S^*} \Gamma(S^* - \xi) f(\xi)\, d\xi = 0,$$

then using the renewal equation, we get the formula (9.3.24). We can check directly that $S^* \geq \bar{s}$. Indeed using

$$1 - \alpha + \alpha \bar{F}(S^*) \leq \frac{1}{\Gamma(S^*)} = \frac{(1-\alpha)(h+p)}{p - (c-h)(1-\alpha)}.$$

Therefore

$$\alpha \bar{F}(S^*) \leq (1-\alpha)\frac{c(1-\alpha) + h\alpha}{p - (c-h)(1-\alpha)}.$$

But, from the assumption (9.3.16), we deduce

$$\frac{1-\alpha}{p - (c-h)(1-\alpha)} \leq \frac{1}{h+p}.$$

From the expression of $\bar{F}(\bar{s})$, we obtain $\bar{F}(S^*) \leq \bar{F}(\bar{s})$, and the result follows. □

Finally we can check the $K-$convexity property

Proposition 9.6. *The function $u(x)$ is K-convex.*

Proof. This is proven as in Theorem 9.6. One proves inductively that the increasing sequence is $K-$convex, and thus this property holds for the limit. □

Remark. If one makes the technical assumption (9.2.33), then $Eu(x-D)$ is continuous, and the s, S property of the optimal feedback can be proven as a consequence of the $K-$convexity theory, as in the proof of Theorem 9.6.

Finally, consider the situation when

$$c > \frac{p\alpha}{1-\alpha}, \tag{9.3.25}$$

then the function $H_s(x)$ is always increasing in x. Therefore $S(s) = s, \forall s$. It follows that the optimal feedback is $\hat{v}(x) = 0$.

9.3.3. PROBABILISTIC APPROACH. As we have done in the case without backlog, we are going to evaluate directly the cost corresponding to an arbitrary s, S policy. We have the following

Theorem 9.12. *Let $V_{s,S}$ be the control corresponding to an s, S policy, defined by the feedback*

$$V_{s,S} = \begin{cases} S - x, & \text{if } x \leq s \\ 0, & \text{if } x > s \end{cases} \tag{9.3.26}$$

The corresponding cost is: For $x \leq s$

$$J_x(V_{s,S}) = hx^+ + px^- - cx + \frac{g(s)}{1-\alpha} + \frac{K + H_s(S)}{(1-\alpha)\Gamma(S-s)}. \tag{9.3.27}$$

For $x > s$, one has

$$\begin{aligned} J_x(V_{s,S}) = hx^+ + px^- - cx + H_s(x) + \frac{g(s)}{1-\alpha} \\ +(1-(1-\alpha)\Gamma(x-s))\frac{K + H_s(S)}{(1-\alpha)\Gamma(S-s)}. \end{aligned} \tag{9.3.28}$$

This function is continuous except for $x = s$.

Proof. The formulas are identical to the case without backlog, however the functions g, H are those corresponding to the present case, see (9.3.11), (9.3.13). The number s is real, and to simplify we take $S > 0$ and $\geq s$.

Let $V_{s,S} = (v_1, \cdots, v_n \cdots)$ and $y_1, y_2, \cdots, y_n \cdots$ be the corresponding evolution of stocks. We have $y_1 = x$. We begin by considering the case

I- $x \leq s \leq 0$:

We then have $v_1 = S - x$. We set $\tau_1 = 1$ and we define the sequence of stopping times

$$\tau_{k+1} = \tau_k + 1 + \inf\{n \geq 0 \mid D_{\tau_k} + \cdots + D_{\tau_k+n} \geq S - s\}. \tag{9.3.29}$$

Since $S - s \geq S$, we will also need another sequence of stopping times

$$\theta_{k+1} = \tau_k + 1 + \inf\{n \geq 0 \mid D_{\tau_k} + \cdots + D_{\tau_k+n} \geq S\}, \tag{9.3.30}$$

and we have $\tau_k + 1 \leq \theta_{k+1} \leq \tau_{k+1}$.

From Lemma 9.5, simply changing $S - s$ into S, we see that for $k \geq 2$, θ_k is a stopping time with respect to the filtration $\mathcal{F}^{n-1}$. Moreover $\theta_{k+1} - \tau_k - 1$ and $D_{\tau_k}, \cdots, D_{\tau_k+n}, n \geq 0$, are mutually independent of $\mathcal{F}^{\tau_k}$. The ordering times of the control $V_{s,S}$ are the times τ_k. The inventory is then

$$y_{\tau_k} = S - (D_{\tau_{k-1}} + \cdots + D_{\tau_k - 1}) \leq s, \forall k \geq 2$$

and the order is

$$v_{\tau_k} = S - y_{\tau_k}, \forall k \geq 2$$

Let us consider the inventory between the times τ_k+1 and $\tau_{k+1}-1$, if $\tau_{k+1}-2 \geq \tau_k$. If $\tau_{k+1}-1=\tau_k$, we have naturally $y_{\tau_k+1}=y_{\tau_{k+1}}$. Otherwise, it is important to consider the position of θ_{k+1}. If $\theta_{k+1}=\tau_k+1$, then the first inventory y_{τ_k+1} is negative and all the inventories till $\tau_{k+1}-1$ are negative. If $\tau_{k+1}-1 \geq \theta_{k+1} \geq \tau_k+2$, then we have

$$\begin{aligned} y_{\tau_k+1} &= S - D_{\tau_k} > 0 \\ &\cdots \\ y_{\theta_{k+1}-1} &= S-(D_{\tau_k}+\cdots+D_{\theta_{k+1}-2}) > 0 \\ s < y_{\theta_{k+1}} &= S-(D_{\tau_k}+\cdots+D_{\theta_{k+1}-1}) \leq 0 \\ s < y_{\tau_{k+1}-1} &= S-(D_{\tau_k}+\cdots+D_{\tau_{k+1}-2}) \leq 0 \end{aligned}$$

If $\theta_{k+1}=\tau_{k+1}$, the inventory is positive from τ_k+1 to $\tau_{k+1}-1$.

Therefore we can write

$$J_x(V_{s,S}) = \sum_{k=1}^{\infty} E\alpha^{\tau_k-1}(K+cS-(c+p))y_{\tau_k}) + \sum_{k=1}^{\infty} \mathbb{1}_{\tau_{k+1}\geq\tau_k+2} \sum_{n=\tau_k+1}^{\tau_{k+1}-1} \alpha^{n-1}(hy_n^+ + py_n^-).$$

We also can write

$$\begin{aligned} J_x(V_{s,S}) &= K+cS-(p+c)x \\ &+\sum_{k=2}^{\infty} E\alpha^{\tau_k-1}(K+cS-cy_{\tau_k}) \\ &+(h+p)E\sum_{k=1}^{\infty} \mathbb{1}_{\theta_{k+1}\geq\tau_k+2} \sum_{n=\tau_k+1}^{\theta_{k+1}-1} \alpha^{n-1}y_n \\ &-pE\sum_{k=1}^{\infty}\sum_{n=\tau_k+1}^{\tau_{k+1}} \alpha^{n-1}y_n, \end{aligned} \tag{9.3.31}$$

which we write as

$$J_x(V_{s,S}) = K+cS-(p+c)x+Z_1.$$

Let us compute Z_1. We first note that

$$E\sum_{k=1}^{\infty} \mathbb{1}_{\theta_{k+1}\geq\tau_k+2} \sum_{n=\tau_k+1}^{\theta_{k+1}-1} \alpha^{n-1}y_n = E\sum_{k=1}^{\infty}\alpha^{\tau_k-1}\left[S\alpha\frac{1-\alpha^{\theta_{k+1}-\tau_k-1}}{1-\alpha} - \sum_{l=0}^{\theta_{k+1}-\tau_k-1} D_{\tau_k+l}\,\alpha\frac{\alpha^l-\alpha^{\theta_{k+1}-\tau_k-1}}{1-\alpha}\right].$$

Similarly one has

$$E\sum_{k=1}^{\infty}\sum_{n=\tau_k+1}^{\tau_{k+1}} \alpha^{n-1}y_n = E\sum_{k=1}^{\infty}\alpha^{\tau_k-1}\left[S\alpha\frac{1-\alpha\alpha^{\tau_{k+1}-\tau_k-1}}{1-\alpha} - \sum_{l=0}^{\tau_{k+1}-\tau_k-1} D_{\tau_k+l}\,\alpha\frac{\alpha^l-\alpha\alpha^{\tau_{k+1}-\tau_k-1}}{1-\alpha}\right].$$

We also write

$$\sum_{k=2}^{\infty} E\alpha^{\tau_k-1}(K+cS-cy_{\tau_k})=E\sum_{k=1}^{\infty}\alpha^{\tau_k-1}\alpha^{\tau_{k+1}-\tau_k-1}\left(\alpha K+\alpha c\sum_{l=0}^{\tau_{k+1}-\tau_k-1}D_{\tau_k+l}\right).$$

From the independence properties we can write

$$E\left[S\alpha\frac{1-\alpha^{\theta_{k+1}-\tau_k-1}}{1-\alpha}-\sum_{l=0}^{\theta_{k+1}-\tau_k-1}D_{\tau_k+l}\,\alpha\frac{\alpha^l-\alpha^{\theta_{k+1}-\tau_k-1}}{1-\alpha}|\mathcal{F}^{\tau_k}\right]$$
$$=E\left[S\alpha\frac{1-\alpha^{\tilde\theta}}{1-\alpha}-\sum_{l=0}^{\tilde\theta}D_{1+l}\frac{\alpha^l-\alpha^{\tilde\theta}}{1-\alpha}\right],$$

where $\tilde\theta=\theta_2-2$. Similarly

$$E\left[S\alpha\frac{1-\alpha\alpha^{\tau_{k+1}-\tau_k-1}}{1-\alpha}-\sum_{l=0}^{\tau_{k+1}-\tau_k-1}D_{\tau_k+l}\,\alpha\frac{\alpha^l-\alpha\alpha^{\tau_{k+1}-\tau_k-1}}{1-\alpha}|\mathcal{F}^{\tau_k}\right]$$
$$=E\left[S\alpha\frac{1-\alpha\alpha^{\theta}}{1-\alpha}-\sum_{l=0}^{\theta}D_{1+l}\,\alpha\frac{\alpha^l-\alpha\alpha^{\theta}}{1-\alpha}\right],$$

where $\theta=\tau_2-2$. Finally

$$E\left[\alpha^{\tau_{k+1}-\tau_k-1}\left(\alpha K+\alpha c\sum_{l=0}^{\tau_{k+1}-\tau_k-1}D_{\tau_k+l}\right)\Big|\mathcal{F}^{\tau_k}\right]=E\left[\alpha^{\theta}\left(\alpha K+\alpha c\sum_{l=0}^{\theta}D_{1+l}\right)\right].$$

So we have

$$Z_1=EX_1\,E\sum_{k=1}^{\infty}\alpha^{\tau_k-1},$$

and

$$E\sum_{k=1}^{\infty}\alpha^{\tau_k-1}=\sum_{k=1}^{\infty}(E\alpha\alpha^{\theta})^{k-1},$$

and from Lemma 9.6

$$E\sum_{k=1}^{\infty}\alpha^{\tau_k-1}=\frac{1}{(1-\alpha)\Gamma(S-s)}.$$

Moreover

$$X_1=\alpha^{\theta}\left(\alpha K+\alpha c\sum_{l=0}^{\theta}D_{1+l}\right)+(h+p)\left(S\alpha\frac{1-\alpha^{\tilde\theta}}{1-\alpha}-\sum_{l=0}^{\tilde\theta}D_{1+l}\frac{\alpha^l-\alpha^{\tilde\theta}}{1-\alpha}\right)$$
$$-p\left(S\alpha\frac{1-\alpha\alpha^{\theta}}{1-\alpha}-\sum_{l=0}^{\theta}D_{1+l}\,\alpha\frac{\alpha^l-\alpha\alpha^{\theta}}{1-\alpha}\right).$$

We recall

$$E\sum_{l=0}^{\theta}\alpha^l D_{1+l}=\bar{D}\Gamma(S-s),$$

and

$$E\alpha^\theta\sum_{l=0}^{\theta}D_{1+l}=\bar{D}\Gamma(S-s)+\frac{1-\alpha}{\alpha}\left(\int_0^{S-s}\Gamma(x)dx-(S-s)\Gamma(S-s)\right).$$

Therefore, collecting results we obtain

$$\begin{aligned}EX_1&=K(1-(1-\alpha)\Gamma(S-s))-cS(1-\alpha)\Gamma(S-s)\\&+\Gamma(S-s)(\alpha c\bar{D}+sc(1-\alpha)+p\alpha\bar{D}-p\alpha s)\\&+(p+h)\left(\int_0^S\Gamma(\xi)d\xi-S\right)+(c(1-\alpha)-p\alpha)\int_0^{S-s}\Gamma(\xi)d\xi.\end{aligned}$$

Note that

$$g(s)=\alpha c\bar{D}+sc(1-\alpha)+p\alpha\bar{D}-p\alpha s.$$

Moreover from the integral equation of Γ, (9.2.19) we have by integration

$$\begin{aligned}\Gamma(S)-S&=\alpha\int_0^S\Gamma(S-\xi)F(\xi)d\xi\\&=\alpha\int_s^S\Gamma(S-\xi)F(\xi)d\xi.\end{aligned}$$

since $s\leq 0$ and $F(\xi)=0$, for $\xi\leq 0$. Also

$$\int_0^{S-s}\Gamma(\xi)d\xi=\int_s^S\Gamma(S-\xi)d\xi.$$

Therefore

$$\begin{aligned}&(p+h)(\int_0^S\Gamma(\xi)d\xi-S)+(c(1-\alpha)-p\alpha)\int_0^{S-s}\Gamma(\xi)d\xi\\&=\int_s^S\Gamma(S-\xi)[c(1-\alpha)-p\alpha+\alpha(p+h)F(\xi)]d\xi\\&=\int_s^S\Gamma(S-\xi)\mu(\xi)d\xi\\&=H(S).\end{aligned}$$

Hence

$$EX_1=K(1-(1-\alpha)\Gamma(S-s))-cS(1-\alpha)\Gamma(S-s)+\Gamma(S-s)g(s)+H(S).$$

It follows that

$$Z_1=\frac{K+H(S)}{(1-\alpha)\Gamma(S-s)}+\frac{g(s)}{1-\alpha}-(K+cS),$$

which implies

$$J_x(V_{s,S})=-(p+c)x+\frac{g(s)}{1-\alpha}+\frac{K+H(S)}{(1-\alpha)\Gamma(S-s)},$$

and thus (9.3.27) is proved in this case.

We now consider the case

II- $s \geq 0,\ x \leq s$**:**

The stopping times θ_k not relevant in this situation. However $y(\tau_k)$ can be positive or negative. We can write

$$
\begin{aligned}
J_x(V_{s,S}) &= K + c(S-x) + hx^+ + px^- \\
&+ \sum_{k=1}^{\infty} E\alpha^{\tau_{k+1}-1}(K + c(S - y_{\tau_{k+1}}) + hy^+_{\tau_{k+1}} + py^-_{\tau_{k+1}}) \\
&+ hE \sum_{k=1}^{\infty} 1\!\!1_{\tau_{k+1} \geq \tau_k + 2} \sum_{n=\tau_k+1}^{\tau_{k+1}-1} \alpha^{n-1} y_n.
\end{aligned}
$$

Operating as before we get

$$
\begin{aligned}
\sum_{k=1}^{\infty} 1\!\!1_{\tau_{k+1} \geq \tau_k + 2} \sum_{n=\tau_k+1}^{\tau_{k+1}-1} \alpha^{n-1} y_n &= \sum_{k=1}^{\infty} \alpha^{\tau_k - 1} \left[S\alpha \frac{1 - \alpha^{\tau_{k+1} - \tau_k - 1}}{1-\alpha} \right. \\
&\left. - \sum_{l=0}^{\tau_{k+1}-\tau_k-1} D_{\tau_k + l}\, \alpha \frac{\alpha^l - \alpha^{\tau_{k+1}-\tau_k-1}}{1-\alpha} \right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \alpha^{\tau_{k+1}-1}(K + c(S - y_{\tau_{k+1}}) + hy^+_{\tau_{k+1}} + py^-_{\tau_{k+1}}) = \\
&\quad \sum_{k=1}^{\infty} \alpha^{\tau_k - 1} \alpha^{\tau_{k+1}-\tau_k-1} \left[\alpha K + \alpha c \sum_{l=0}^{\tau_{k+1}-\tau_k-1} D_{\tau_k+l} \right. \\
&\quad \left. + \alpha h \left(S - \sum_{l=0}^{\tau_{k+1}-\tau_k-1} D_{\tau_k+l} \right)^+ + \alpha p \left(S - \sum_{l=0}^{\tau_{k+1}-\tau_k-1} D_{\tau_k+l} \right)^- \right].
\end{aligned}
$$

It follows that

$$
J_x(V_{s,S}) = K + c(S-x) + hx^+ + px^- + Z_2,
$$

with

$$
Z_2 = \frac{EX_2}{(1-\alpha)\Gamma(S-s)},
$$

and

$$
X_2 = \alpha^{\theta} \Phi \left(\sum_{l=0}^{\theta} D_{1+l} \right) + h \left[S\alpha \frac{1-\alpha^{\theta}}{1-\alpha} - \sum_{l=0}^{\theta} D_{1+l}\, \alpha \frac{\alpha^l - \alpha^{\theta}}{1-\alpha} \right],
$$

with

$$
\Phi(\xi) = \alpha K + \alpha c \xi + \alpha h (S-\xi)^+ + \alpha p (S-\xi)^-.
$$

To compute EX_2 we need to use the formula (9.2.47) of Lemma 9.6. After rearrangements we obtain

$$
\begin{aligned}
E\alpha^{\theta} \Phi \left(\sum_{l=0}^{\theta} D_{1+l} \right) &= K(1 - (1-\alpha)\Gamma(S-s)) - cS(1-\alpha)\Gamma(S-s) \\
&- hs\Gamma(S-s) + \Gamma(S-s)g(s) + H(S) + hS - h \int_s^S \Gamma(S-\xi)d\xi,
\end{aligned}
$$

and from previous results

$$EhS\left[\alpha\frac{1-\alpha^{\theta}}{1-\alpha}-\sum_{l=0}^{\theta}D_{1+l}\,\alpha\frac{\alpha^{l}-\alpha^{\theta}}{1-\alpha}\right]=sh\Gamma(S-s)-hS+h\int_{s}^{S}\Gamma(S-\xi)d\xi.$$

Therefore

$$EX_2=K+H(S)-(K+cS)(1-\alpha)\Gamma(S-s)+\Gamma(S-s)g(s)=EX_1.$$

Hence $Z_2=Z_1$ and we have again ((9.3.27).

We next turn to the case

III- $x>0>s$

Now $\tau_1\geq 2$. We have the definition

$$\tau_1=2+\inf\{n\geq 0|D_1+\cdots+D_{1+n}\geq x-s\},$$

and we need to introduce

$$\theta_1=2+\inf\{n\geq 0|D_1+\cdots+D_{1+n}\geq x\},$$

and $\theta_1\leq\tau_1$. We may write

$$J_x(V_{s,S})=hx+E1\!\!1_{\tau_1\geq 3}\sum_{j=0}^{\tau_1-3}\alpha^{j+1}(hy_{j+2}^{+}+py_{j+2}^{-})+E\alpha^{\tau_1-1}(K+cS-(c+p)y_{\tau_1})+\tilde{Z}_1.$$

The number $\tilde{Z}_1$ stems from the same definition as Z_1, see (9.3.31), except for the fact that τ_1 is no more 1. Making use of independence properties one can check that

$$\tilde{Z}_1=Z_1E\alpha^{\tau_1-1}.$$

Now τ_1-2 has the same distribution as θ, except that S must be relaxed for x. It follows that

$$E\alpha^{\tau_1-1}=1-(1-\alpha)\Gamma(x-s). \tag{9.3.32}$$

We next write

$$\begin{aligned}&E1\!\!1_{\tau_1\geq 3}\sum_{j=0}^{\tau_1-3}\alpha^{j+1}(hy_{j+2}^{+}+py_{j+2}^{-})\\&=(h+p)E1\!\!1_{\theta_1\geq 3}\sum_{j=0}^{\theta_1-3}\alpha^{j+1}y_{j+2}-pE1\!\!1_{\tau_1\geq 3}\sum_{j=0}^{\tau_1-3}\alpha^{j+1}y_{j+2}\\&=(h+p)E\left[x\frac{\alpha-\alpha\alpha^{\theta_1-2}}{1-\alpha}-\sum_{l=0}^{\theta_1-2}D_{1+l}\alpha\frac{\alpha^{l}-\alpha^{\theta_1-2}}{1-\alpha}\right]\\&\quad-pE\left[x\frac{\alpha-\alpha\alpha^{\tau_1-2}}{1-\alpha}-\sum_{l=0}^{\tau_1-2}D_{1+l}\alpha\frac{\alpha^{l}-\alpha^{\tau_1-2}}{1-\alpha}\right].\end{aligned}$$

Then

$$E\alpha^{\tau_1-1}(K+cS-(c+p)y_{\tau_1})=E\alpha^{\tau_1-1}\left[K+cS-(p+c)x+(p+c)\sum_{l=0}^{\tau_1-2}D_{1+l}\right].$$

We obtain

$$J_x(V_{s,S}) = Y_1 + (1-(1-\alpha)\Gamma(x-s))\left[\frac{K+H(S)}{(1-\alpha)\Gamma(S-s)} - (K+cS) + \frac{g(s)}{1-\alpha}\right],$$

with

$$\begin{aligned} Y_1 = hx + (h+p)E\left[x\frac{\alpha-\alpha\alpha^{\theta_1-2}}{1-\alpha} - \sum_{l=0}^{\theta_1-2} D_{1+l}\alpha\frac{\alpha^l-\alpha^{\theta_1-2}}{1-\alpha}\right] \\ -pE\left[x\frac{\alpha-\alpha\alpha^{\tau_1-2}}{1-\alpha} - \sum_{l=0}^{\tau_1-2} D_{1+l}\alpha\frac{\alpha^l-\alpha^{\tau_1-2}}{1-\alpha}\right] \\ +E\alpha^{\tau_1-1}\left[K+cS-(p+c)x+(p+c)\sum_{l=0}^{\tau_1-2} D_{1+l}\right]. \end{aligned}$$

The expressions with $\tau_1 - 2$ and $\theta_1 - 2$ are expressed as similar ones before with θ and $\tilde{\theta}$, except that one replaces S with x. Therefore

$$\begin{aligned} Y_1 = hx - cx + (K+cS)(1-(1-\alpha)\Gamma(x-s)) + g(s)\Gamma(x-s) \\ +(h+p)\left(\int_0^x \Gamma(\xi)d\xi - x\right) + (c(1-\alpha)-p\alpha)\int_0^{x-s}\Gamma(\xi)d\xi. \end{aligned}$$

We then notice that

$$(h+p)\left(\int_0^x \Gamma(\xi)d\xi - x\right) + (c(1-\alpha)-p\alpha)\int_0^{x-s}\Gamma(\xi)d\xi = H(x),$$

using the fact that $s < 0$, as we have done for $H(S)$ earlier. Collecting results we obtain (9.3.28).

We turn to the case

IV- $0 > x > s.$

We then have

$$J_x(V_{s,S}) = -px - pE\sum_{j=0}^{\tau_1-2}\alpha^{j+1}y_{j+2} + E\alpha^{\tau_1-1}(K+cS-cy_{\tau_1}) + \tilde{Z}_1.$$

We see again that

$$\begin{aligned} J_x(V_{s,S}) = -px - cx + H(x) + g(s)\Gamma(x-s) \\ +(K+cS)(1-(1-\alpha)\Gamma(x-s)) + \tilde{Z}_1, \end{aligned}$$

and we again obtain (9.3.28).

It remains to consider the last case

V- $x > s > 0.$

With similar techniques we check that

$$J_x(V_{s,S}) = Y_2 + \tilde{Z}_2,$$

with

$$\begin{aligned}
\tilde{Z}_2 &= Z_2(1-(1-\alpha)\Gamma(x-s)) \\
&= Z_1(1-(1-\alpha)\Gamma(x-s)) \\
&= (1-(1-\alpha)\Gamma(x-s))\left[\frac{K+H(S)}{(1-\alpha)\Gamma(S-s)} - (K+cS) + \frac{g(s)}{1-\alpha}\right],
\end{aligned}$$

and

$$Y_2 = hx + hE\sum_{j=0}^{\tau_1-2}\alpha^{j+1}y_{j+2} + E\alpha^{\tau_1-1}(K+cS-cy_{\tau_1}) + (h+p)E\alpha^{\tau_1-1}y_{\tau_1}^-,$$

so

$$\begin{aligned}
Y_2 = hx + hE&\left[x\frac{\alpha-\alpha^2\alpha^{\tau_1-2}}{1-\alpha} - \sum_{l=0}^{\tau_1-2}D_{1+l}\alpha\frac{\alpha^l-\alpha\alpha^{\tau_1-2}}{1-\alpha}\right] \\
&+E\alpha^{\tau_1-2}\Phi_x\left(\sum_{l=0}^{\tau_1-2}D_{1+l}\right),
\end{aligned}$$

in which

$$\Phi_x(\xi) = \alpha K + \alpha c(S-x) + \alpha c\xi + \alpha(h+p)(x-\xi)^-.$$

Note that x is simply a parameter here (playing the role of S before). By using familiar techniques we can check that

$$Y_2 = hx - cx + H(x) + (K+cS)(1-(1-\alpha)\Gamma(x-s)) + \Gamma(x-s)g(s),$$

and (9.3.28) is verified. This completes the proof. □

We have the same situation in the backlog case as in the case without backlog. From formulas (9.3.27), (9.3.28) we see that one can optimize S independently of x. It is sufficient to minimize the function

$$Z_s(x) = \frac{K+H_s(x)}{\Gamma(x-s)},$$

for $x \geq s$. It is easy to check that there is a minimum, and we can always take the smallest minimum. This is identical to the case without backlog. So in fact, there is only one remaining choice, the value of s. We can then define the function $u_s(x)$ by the formulas

(9.3.33)
$$\begin{aligned}
u_s(x) = -cx + hx^+ + px^- + H_s(x) + \frac{g(s)}{1-\alpha} \\
+\frac{(1-(1-\alpha)\Gamma(x-s))}{1-\alpha}\inf_{\xi\geq s}\frac{K+H_s(\xi)}{\Gamma(\xi-s)}, \quad \forall x > s
\end{aligned}$$

(9.3.34)
$$\begin{aligned}
u_s(x) = (h-c)x + \frac{g(s)}{1-\alpha} + \\
+\frac{1}{1-\alpha}\inf_{\xi\geq s}\frac{K+H_s(\xi)}{\Gamma(\xi-s)}, \quad \forall x \leq s
\end{aligned}$$

There is a discontinuity for $x = s$, unless

(9.3.35)
$$\inf_{\xi\geq s}\frac{K+H_s(\xi)}{\Gamma(\xi-s)} = 0,$$

which is equivalent to the condition $K + \inf_{\xi\geq s} H_s(\xi) = 0$. We recover the pair s, S obtained by the analytic theory of Impulse Control, Theorem 9.11. So the choice of s is dictated by a continuity condition of the cost function, and not by an optimization (unlike the choice of S). As for the case without backlog, we have shown in Theorem 9.11, that the function defined by this s,S control policy is the value function and is solution of the Bellman equation.

9.3.4. INVARIANT MEASURE. We have already obtained the invariant measure in section 3.6.2 of Chapter 3. We recall the result. The state space is $X=(-\infty,S]$ and the invariant measure has a density with respect to Lebesgue's measure given by

$$m(\eta)=\begin{cases}\frac{f(S-\eta)+\int_s^S z(x)f(x-\eta)dx}{1+\int_s^S z(y)dy}, & \text{if } \eta\leq s\\ \frac{z(\eta)}{1+\int_s^S z(y)dy}, & \text{if } s\leq\eta\leq S\end{cases} \tag{9.3.36}$$

where the function $z(\eta)$ is defined in s,S as the unique solution of the integral equation

$$z(\eta)=f(S-\eta)+\int_\eta^S z(x)f(x-\eta)dx,\quad s\leq\eta\leq S, \tag{9.3.37}$$

which has a unique solution in $B(s,S)$, if $F(S)<1$.

Consider now the objective function $J_x(V_{s,S})$, when we apply a control $V=V_{s,S}$ defined by an s,S policy. The inventory y_n becomes ergodic so we have

$$(1-\alpha)\int_{-\infty}^S J_x(V_{s,S})m_{s,S}(x)dx=\int_{-\infty}^S [hx^+ + px^- + (K+c(S-x))\mathbb{1}_{x\leq s}]m_{s,S}(x)dx.$$

Note that the integrals are well defined, in view of the fact that $\bar{D}$ is finite, see also (9.3.38) below.

Also

$$\begin{aligned}(1-\alpha)\int_{-\infty}^S J_x(V_{s,S})m_{s,S}(x)dx &= \int_{-\infty}^S [(1-\alpha)(hx^+ + px^- - cx)\\ &+(K+c(S-x))\mathbb{1}_{x\leq s}]m_{s,S}(x)dx\\ &+\int_{-\infty}^S [c(1-\alpha)x+\alpha(hx^+ + px^-)]m_{s,S}(x)dx.\end{aligned}$$

We omit the indices s,S to simplify notation. Recall that

$$g(x)=cx(1-\alpha)+\alpha c\bar{D}+\alpha hE(x-D)^+ + \alpha pE(x-D)^-.$$

It follows that

$$\begin{aligned}g(x)\mathbb{1}_{x>s}+g(S)\mathbb{1}_{x\leq s} &= c(1-\alpha)x+\alpha c\bar{D}+c(1-\alpha)(S-x)\mathbb{1}_{x\leq s}\\ &+\alpha hE(x-D)^+\mathbb{1}_{x>s}+\alpha hE(S-D)^+\mathbb{1}_{x\leq s}\\ &+\alpha pE(x-D)^-\mathbb{1}_{x>s}+\alpha pE(S-D)^-\mathbb{1}_{x\leq s}.\end{aligned}$$

On the other hand, from the equation giving $H(x)$, see (9.3.13), we have

$$\begin{aligned}g(x)\mathbb{1}_{x>s}+g(S)\mathbb{1}_{x\leq s} &= H(x)+H(S)\mathbb{1}_{x\leq s}+g(s)\\ &-\alpha[EH(x-D)\mathbb{1}_{x>s}+EH(S-D)\mathbb{1}_{x\leq s}].\end{aligned}$$

Integrating with respect to the invariant measure, and making use of its fundamental property we get

$$\int_{-\infty}^{S}(g(x)\mathbb{1}_{x>s}+g(S)\mathbb{1}_{x\leq s})m(x)dx$$
$$=\alpha c\bar{D}+\int_{-\infty}^{S}[c(1-\alpha)x+\alpha h^{+}+\alpha p^{-}]m(x)dx$$
$$+c(1-\alpha)\int_{-\infty}^{s}(S-x)m(x)dx$$
$$=(1-\alpha)\int_{-\infty}^{S}H(x)m(x)dx+g(s)+H(S)\int_{-\infty}^{S}m(x)dx.$$

Therefore

$$(1-\alpha)\int_{-\infty}^{S}J_x(V)m(x)dx=\int_{-\infty}^{S}(1-\alpha)(hx^{+}+px^{-}-cx+H(x))m(x)dx$$
$$+(K+H(S))\int_{-\infty}^{s}m(x)dx+g(s)+c\alpha\left[\int_{-\infty}^{s}(S-x)m(x)dx-\bar{D}\right].$$

Now we use the property

$$\bar{D}=\int_{-\infty}^{s}(S-x)m(x)dx. \tag{9.3.38}$$

This follows from writing

$$\int_{-\infty}^{S}xm(x)dx=\int_{-\infty}^{S}[\mathbb{1}_{x>s}E(x-D)+\mathbb{1}_{x\leq s}E(S-D)]m(x)dx,$$

which is a consequence of the fundamental property of the invariant measure. Therefore, we obtain

$$\int_{-\infty}^{S}J_x(V)m(x)dx=\int_{-\infty}^{S}(hx^{+}+px^{-}-cx+H(x))m(x)dx$$
$$+\frac{g(s)}{1-\alpha}+\frac{K+H(S)}{1-\alpha}\int_{0}^{s}m(x)dx. \tag{9.3.39}$$

We can show directly formula (9.3.39) from the formulas giving $J_x(V)$, formulas (9.3.27), (9.3.28). One needs to prove that the function $\Gamma(x)$ satisfies the relation

$$\Gamma(S-s)\int_{-\infty}^{s}m(dx)+(1-\alpha)\int_{s}^{S}\Gamma(x-s)m(dx)=1, \tag{9.3.40}$$

which is similar to (9.2.71).

9.3.5. PARTICULAR CASE. Consider the case $f(x)=\beta\exp-\beta x$. We recall that

$$\Gamma(x)=\frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha}\exp-\beta(1-\alpha)x, \tag{9.3.41}$$

then

$$\mu(x)=c(1-\alpha)+\alpha h-\alpha(p+h)\exp-\beta x^{+}, \tag{9.3.42}$$

and

$$g(x)=c(1-\alpha)x+\alpha(hx^{+}+px^{-})+\alpha(c+p)\bar{D}+\alpha\frac{h+p}{\beta}(\exp-\beta x^{+}-1). \tag{9.3.43}$$

The value $\bar{s} > 0$ such that $\mu(\bar{s}) = 0$ is given by

$$\exp -\beta\bar{s} = \frac{c(1-\alpha)+\alpha h}{\alpha(p+h)}, \tag{9.3.44}$$

thanks to the assumption (9.3.16).

Next we have, for $x > s$

$$\begin{aligned}H'(x) &= c(1-\alpha)+\alpha h - \alpha(p+h)\exp -\beta x^{+} \\ &\quad + \frac{\alpha}{1-\alpha}(c(1-\alpha)+\alpha h)[1-\exp -\beta(1-\alpha)(x-s)] \\ &\quad -\alpha^2\beta(p+h)\exp -\beta(1-\alpha)x\left\{\frac{\exp -\beta(1-\alpha)x^{-} - \exp -\beta(1-\alpha)s^{-}}{\beta(1-\alpha)}\right. \\ &\quad \left. - \frac{\exp -\beta\alpha x^{+} - \exp -\beta\alpha s^{+}}{\beta\alpha}\right\}.\end{aligned}$$

For $s \leq \bar{s}$, The number S is uniquely defined by the equation $H'(S) = 0$.

Let us now turn to the invariant measure. The solution of (9.3.37) is $z(\eta) = \beta$ and

$$m(\eta) = \begin{cases} \dfrac{\beta}{1+\beta(S-s)}, & \text{if } s \leq \eta \leq S \\ \dfrac{\beta\exp -\beta(s-\eta)}{1+\beta(S-s)}, & \text{if } \eta \leq s \end{cases} \tag{9.3.45}$$

CHAPTER 10

ERGODIC CONTROL OF INVENTORIES WITH SET UP COST

In this chapter, we shall consider the situations studied in the previous chapter, in the case $\alpha \to 1$.

10.1. DETERMINISTIC CASE

10.1.1. SETTING OF THE PROBLEM. We consider the functional equation

$$u_\alpha(x) = hx + \inf_{v \geq (D-x)^+} [K \mathbb{1}_{v>0} + cv + \alpha u_\alpha(x+v-D)], \tag{10.1.1}$$

in which we have emphasized the dependence in α. We know that the optimal feedback is given by

$$\hat{v}_\alpha(x) = \begin{cases} 0, & \text{if } x \geq D \\ D(\hat{k}_\alpha + 1) - x, & \text{if } x < D \end{cases} \tag{10.1.2}$$

where $\hat{k}_\alpha$ minimizes

$$\lambda_\alpha(k) = \frac{K + c(k+1)D + \alpha h D F_\alpha(k)}{1 - \alpha^{k+1}},$$

where

$$F_\alpha(k) = \frac{\alpha^{k+1} - (k+1)\alpha + k}{(1-\alpha)^2}.$$

We want to study the behavior of $u_\alpha(x)$ and $\hat{v}_\alpha(x)$ as $\alpha \to 1$.

10.1.2. CONVERGENCE. We define the integer $\hat{k}$ which minimizes

$$\lambda_1(k) = \frac{K}{k+1} + hD\frac{k}{2}, \tag{10.1.3}$$

over $k \geq 0$. We next define

$$\rho = cD + \lambda_1(\hat{k}), \tag{10.1.4}$$

then we can state the following convergence result

Theorem 10.1. *We have the properties, as* $\alpha \to 1$

$$\varrho_\alpha = (1-\alpha)u_\alpha(0) \to \rho; \tag{10.1.5}$$

$$z_\alpha(x) = u_\alpha(x) - u_\alpha(0) \to z(x), \tag{10.1.6}$$

with

$$z(x) = \begin{cases} (h-c)x, & \textit{if } x < D \\ hx - \rho + z(x-D), & \textit{if } x \geq D \end{cases} \tag{10.1.7}$$

We also can write

$$z(x)+\rho = hx + \inf_{v\geq (D-x)^+}[K\mathbb{1}_{v>0}+cv+z(x+v-D)], \tag{10.1.8}$$

and the pair $z(x),\rho$ *given by* (10.1.11), (10.1.4) *is the unique solution of* (10.1.8). *Moreover the optimal feedback is*

$$\hat{v}(x) = \begin{cases} 0, & \text{if } x \geq D \\ D(\hat{k}+1)-x, & \text{if} x < D \end{cases} \tag{10.1.9}$$

PROOF. The solution of (10.1.11) is simply

$$\begin{aligned} z(kD+\xi) &= hD\frac{k(k+1)}{2} + \xi h(k+1) - k\rho, \forall 0\leq \xi < D; \qquad (10.1.10)\\ z(((k+1)D)^+) &= hD\frac{(k+1)(k+2)}{2} - (k+1)\rho, \end{aligned}$$

where again $z(((k+1)D)^+)$ represents the limit to the right at point $(k+1)D$. We know that

$$u_\alpha(0) = \lambda(\hat{k}_\alpha),$$

and it is easy to check that

$$(1-\alpha)\lambda_\alpha(k) \to cD + \lambda_1(k), \forall k.$$

This property carries over to the infimum,

$$\hat{k}_\alpha \to \hat{k},$$

which proves (10.1.5). By definition,

$$z_\alpha(x) = \begin{cases} (h-c)x, & \text{if } x < D \\ hx - \rho_\alpha + z_\alpha(x-D), & \text{if } x \geq D \end{cases} \tag{10.1.11}$$

Thanks to (10.1.5), the convergence of $z_\alpha(x)$ to $z(x)$ follows easily. To prove (10.1.8), one looks at (10.1.1), which can be written as

$$z_\alpha(x)+\rho_\alpha = hx + \inf_{v\geq (D-x)^+}[K\mathbb{1}_{v>0}+cv+\alpha z_\alpha(x+v-D)]. \tag{10.1.12}$$

We note the relation

$$\rho_\alpha = K + Dc(\hat{k}_\alpha+1) + \alpha z_\alpha(\hat{k}_\alpha D). \tag{10.1.13}$$

Denote

$$G_\alpha(x,v) = K\mathbb{1}_{v>0} + c(v+x) + \alpha z_\alpha(x+v-D),$$

and

$$G_\alpha(x) = \inf_{v\geq (D-x)^+} G_\alpha(x,v) = G_\alpha(x,\hat{v}_\alpha(x)),$$

and similarly

$$\begin{aligned} G(x,v) &= K\mathbb{1}_{v>0} + c(v+x) + z(x+v-D); \\ G(x) &= \inf_{v\geq (D-x)^+} G(x,v) = G(x,\hat{v}(x)). \end{aligned}$$

We claim that

$$G_\alpha(x) \to G(x). \tag{10.1.14}$$

Indeed, when $x \geq D$, $G_\alpha(x) = G_\alpha(x,0) \to G(x,0) = G(x)$. When $x < D$, then from (10.1.13) and the expression of $\hat{v}_\alpha(x)$

$$G_\alpha(x) = \rho_\alpha \to \rho = G(x).$$

We can pass to the limit in (10.1.12) and obtain (10.1.8) and (10.1.9). The proof has been completed. □

10.1.3. INTERPRETATION. The interpretation of ρ is easy. Consider an inventory evolution

$$y_{n+1} = y_n + v_n - D, \quad y_1 = 0,$$

and use the following policy. Order at time 1, $v_1 = (k+1)D$ (we must put an order otherwise the inventory will become negative, which is forbidden). We then order nothing till the inventory becomes 0. This occurs at time $k+1$. The cost of this policy is

$$K + cD(k+1) + hD\frac{k(k+1)}{2},$$

and the cost per unit of time is

$$\frac{K}{k+1} + cD + \frac{hDk}{2}.$$

So ρ is the minimum average cost. Concerning $z(x)$, we can refer to (10.1.8) and interpret it as a Bellman equation. Considering the evolution

$$\begin{aligned} y_{n+1} &= y_n + v_n - D, \quad y_1 = x \\ v_n &\geq (D - y_n)^+ \end{aligned}$$

and the cost functional

(10.1.15) $$J_x(V) = \sum_{n=1}^{\infty} [K \mathbb{1}_{v_n > 0} + cv_n + hy_n - \rho],$$

then one has

$$z(x) = \inf_V J_x(V).$$

Remark. If we consider the equation (10.1.3) which defines $\lambda_1(k)$ and make the change of variable $q = Dk$, we obtain, making the approximation $D(k+1) \sim q$,

$$\lambda_1(k) = \tilde{\lambda}_1(q) = \frac{KD}{q} + \frac{hq}{2},$$

and we recover the cost which has led to the EOQ formula, see section 2.2.1.

10.2. ERGODIC INVENTORY CONTROL WITH FIXED COST AND NO SHORTAGE

10.2.1. STATEMENT OF THE PROBLEM. We consider the Bellman equation

(10.2.1) $$u_\alpha(x) = (h-c)x + \inf_{\eta \geq x}\{K\mathbb{1}_{\eta > x} + c\eta + pE((\eta - D)^-) + \alpha E u_\alpha((\eta - D)^+)\},$$

studied in section 9.2. Define

(10.2.2) $$g_\alpha(x) = (p - \alpha(c-h))E(x-D)^+ - (p-c)x + p\bar{D},$$

and

(10.2.3) $$g_{\alpha,s}(x) = (g_\alpha(x) - g_\alpha(s))\mathbb{1}_{x \geq s},$$

(10.2.4) $$H_{\alpha,s}(x) = \sum_{n=1}^{\infty} \alpha^{n-1} g_{\alpha,s} \star f^{\star(n-1)}(x).$$

Let $\mu_\alpha(x) = g'_\alpha(x)$, then

$$H_{\alpha,s}(x) = \int_s^x \Gamma_\alpha(x-\xi)\mu_\alpha(\xi)d\xi, \tag{10.2.5}$$

where $\Gamma_\alpha(x)$ is the solution of the renewal equation

$$\Gamma_\alpha(x) = 1 + \alpha\int_0^x f(x-\xi)\Gamma_\alpha(\xi)\,d\xi. \tag{10.2.6}$$

We define the function $S_\alpha(s)$ such that

$$\inf_{\eta \geq s} H_{\alpha,s}(\eta) = H_{\alpha,s}(S_\alpha(s)), \tag{10.2.7}$$

and the pair s_α, S_α is defined by the relations

$$\inf_{\eta \geq s_\alpha} H_{\alpha,s_\alpha}(\eta) + K = 0, \quad S_\alpha = S_\alpha(s_\alpha). \tag{10.2.8}$$

We know that

$$0 \leq s_\alpha \leq \bar{s}_\alpha, \tag{10.2.9}$$

with

$$c(1-\alpha) + \alpha h - (p - \alpha(c-h))\bar{F}(\bar{s}_\alpha) = 0. \tag{10.2.10}$$

Finally

$$u_\alpha(x) = H_{\alpha,s_\alpha}(x) + (h-c)x + \frac{g_\alpha(s_\alpha)}{1-\alpha}. \tag{10.2.11}$$

We want to study the behavior of $u_\alpha(x)$ as $\alpha \uparrow 1$.

10.2.2. CONVERGENCE. We shall make the assumption

$$F(x) < 1, \forall x. \tag{10.2.12}$$

Thanks to the assumption (10.2.12) the function $\Gamma_\alpha(x)$ converges to $\Gamma(x)$ where

$$\Gamma(x) = \sum_{n=0}^{\infty} F^{(n)}(x).$$

We note that

$$\Gamma(x) \leq \frac{1}{1-F(x)},$$

and $\Gamma(x)$ is the solution of the renewal equation

$$\Gamma(x) = 1 + \int_0^x f(x-\xi)\Gamma(\xi)d\xi. \tag{10.2.13}$$

We have next

$$\mu_\alpha(x) \to \mu(x) = h - (p-c+h)\bar{F}(x). \tag{10.2.14}$$

Similarly

$$\begin{aligned} g_\alpha(x) \quad \to \quad g(x) &= hx + p\bar{D} - (p-c+h)\int_0^x \bar{F}(\xi)d\xi \\ &= hx + (c-h)\bar{D} + (p-c+h)E(x-D)^- \end{aligned} \tag{10.2.15}$$

The function

$$H_{\alpha,s}(x) \to H_s(x) = \int_s^x \Gamma(x-\xi)\mu(\xi)d\xi. \tag{10.2.16}$$

Assuming (9.2.12) we define $\bar{s}$ such that $\mu(\bar{s}) = 0$, and $\mu(s) < 0, \forall 0 \leq s < \bar{s}$, $\mu(s) > 0, \forall s > \bar{s}$.

Lemma 10.1. *The minimum of $H_s(x)$ for $x \geq s$ is attained in points strictly larger than $\bar{s}$. Moreover $\min_{x \geq s} H_s(x) < 0$.*

PROOF. Unlike what has been done for $H_{\alpha,s}(x)$, we cannot rely on the fact that $H_{\alpha,s}(x) \to +\infty$ as $x \to +\infty$. This is because $\Gamma(x)$ is not a priori bounded like $\Gamma_\alpha(x)$. For $s < \bar{s}$, we can assume that $s \leq \bar{s}_\alpha$. Therefore we have

$$s \leq \bar{s}_\alpha < S_\alpha(s) < S_\alpha(0).$$

Recalling that

$$\Gamma_\alpha(S_\alpha(0)) = \frac{p - \alpha(c-h)}{p - \alpha(p-h)},$$

we can assert that $S_\alpha(0) \to S(0)$ solution of

$$\Gamma(S(0)) = \frac{p-c+h}{h}. \tag{10.2.17}$$

Therefore we can extract a subsequence of $S_\alpha(s)$ which converges to S_s^*. Clearly

$$s < \bar{s} \leq S_s^* \leq S(0).$$

Clearly $H_{\alpha,s}(S_\alpha(s)) \to H_s(S_s^*)$. Since $H_{\alpha,s}(S_\alpha(s)) \leq H_{\alpha,s}(x), \forall x \geq s$, we get also $H_s(S_s^*) \leq H_s(x), \forall x \geq s$. Therefore $H_s(S_s^*) = \inf_{x \geq s} H_s(x)$. So the infimum is attained. Since $H_s'(s+) < 0$, the infimum is not attained in s and is strictly negative. Note that S_s^* is solution of the equation

$$\Gamma(S-s)\mu(s) + (p-c+h)\int_s^{S_s^*} \Gamma(S-\xi)f(\xi)\,d\xi = 0, \tag{10.2.18}$$

and thus we cannot have $S_s^* = \bar{s}$, if $s < \bar{s}$. □

We then define in a unique way the smallest minimum, called $S(s)$ and $S(s) > \bar{s}$. Moreover $S(s) = s, \forall s \geq \bar{s}$.

We have again

$$\frac{d}{ds} H_s(S(s)) = -\Gamma(S(s) - s)\mu(s) > 0, 0 \leq s < \bar{s}.$$

The function $H_s(S(s))$ increases from $H_0(S(0))$ to 0 when s goes from 0 to $\bar{s}$.

Assume

$$H_0(S(0)) < -K, \tag{10.2.19}$$

then there exists a unique s such

$$H_s(S(s)) = -K. \tag{10.2.20}$$

Theorem 10.2. *We assume* (9.2.12), (10.2.12), (10.2.19), *then we have*

$$s_\alpha \to s, \quad S_\alpha \to S; \tag{10.2.21}$$

$$u_\alpha(x) - \frac{g_\alpha(s_\alpha)}{1-\alpha} \to u(x) = H(x) + (h-c)x. \tag{10.2.22}$$

Set $\rho = g(s)$, then $\rho > 0$ and one has

$$u(x) = (h-c)x - \rho + \inf_{\eta \geq x}[K 1\!\!1_{\eta > x} + c\eta + pE(\eta - D)^- + Eu((\eta - D)^+)]. \tag{10.2.23}$$

PROOF. We use again the fact that $S_\alpha(s)$ is decreasing, see Proposition 9.3. Hence $S_\alpha = S_\alpha(s_\alpha) \leq S_\alpha(0)$, which remains bounded. We have $S_\alpha(0) \to S(0) = S_0$ (to shorten the notation). Also $H_{\alpha,0}(S_\alpha(0)) \to H_0(S_0)$. Therefore, from the assumption (10.2.19), we can assume that, for α sufficiently close to 1, $H_{\alpha,0}(S_\alpha(0)) < -K$. Therefore the pair s_α,S_α is well defined, and these numbers remain bounded. So we can extract a subsequence which converges to s^*, S^*. From the expression of $H_{\alpha,s}(x)$, we easily deduce that $H_{\alpha,s_\alpha}(S_\alpha) = -K \to H_{s^*}(S^*)$. From this one obtains that $s^* = s$, $S^* = S$, defined by equations (10.2.20), (10.2.18).

Note that

$$g(x) = hx + p\int_x^\infty \bar{F}(\xi)\,d\xi + (c-h)\int_0^x \bar{F}(\xi)\,d\xi, \tag{10.2.24}$$

so it is a positive function, thanks to the assumption (9.2.12). Hence $\rho > 0$. Clearly

$$(1-\alpha)u_\alpha(x) - g_\alpha(s_\alpha) \to 0, \text{ as } \alpha \uparrow 1,$$

hence

$$(1-\alpha)u_\alpha(x) \to \rho = g(s).$$

We recall that we have also

$$H(x) + \rho = g(x) + EH((x-D)^+),\ \forall x \geq s;$$

$$H(x) = 0, \forall x \leq s,$$

where $H(x) = H_s(x)$, defined by (10.2.16), for the specific s, limit of s_α.

Let us check that these conditions can be summarized into

$$H(x) + \rho = +\inf_{\eta \geq x}[K\mathbb{1}_{\eta > x} + g(\eta) + EH((\eta - D)^+)]. \tag{10.2.25}$$

If (10.2.25) is satisfied, then the equation for u (10.2.23) follows immediately, using the definition of $g(x)$.

Let us check (10.2.25). Take first $x \leq s$. Then $H(x) = 0$. If we express the right hand side of (10.2.25), then we have to consider the case $\eta = x$ and $\eta > x$. For $\eta = x$ we have

$$-\rho + g(x) + EH((x-D)^+) = g(x) - g(s) > 0,$$

since $x < s < \bar{s}$. On the other hand for $x < \eta < s$ we have

$$-\rho + K + g(\eta) + EH((\eta - D)^+) = -\rho + K + g(\eta),$$

which is decreasing. Therefore

$$\begin{aligned} &-\rho + K + \inf_{\eta > x}[g(\eta) + EH((\eta - D)^+)] \\ &\quad = -\rho + K + \inf_{\eta \geq s}[g(\eta) + EH((\eta - D)^+)] \\ &\quad = -\rho + K + \inf_{\eta \geq s}[H(\eta) + g(s)] = 0. \end{aligned}$$

Hence (10.2.25) is verified for $x \leq s$ Assume $x > s$. Then

$$-\rho + \inf_{\eta \geq x}[K\mathbb{1}_{\eta > x} + g(\eta) + EH((\eta - D)^+)] = \inf_{\eta \geq x}[K\mathbb{1}_{\eta > x} + H(\eta)]. \tag{10.2.26}$$

But, from (12.5.3) we have, denoting $H_\alpha(x) = H_{\alpha,s_\alpha}(x)$

$$H_\alpha(x) \leq K + \inf_{\eta \geq x} H_\alpha(\eta), \forall x \geq s_\alpha.$$

But $H_{\alpha,s_\alpha}(x) \to H(x)$, We pass to the limit and obtain

$$H(x) \leq K + \inf_{\eta \geq x} H(\eta), \forall x \geq s$$

and the right hand side of (10.2.26) is $H(x)$. This completes the proof of the Theorem. □

10.2.3. PROBABILISTIC INTERPRETATION. We give now the interpretation of ρ and of the solution $u(x)$ of (10.2.23). We first associate to an s, S policy an invariant measure $m_{s,S}(dx)$. Moreover the infimum in equations (10.2.23), (10.2.25) is attained by a feedback $\hat{v}(x)$ associated to an s, S policy. We still denote it s, S to save notation.

It is convenient to notice that

$$l(x,v) = p\bar{D} + (h-p)x + \mathbb{1}_{v>0}(K + (c-p)v) + pE(x+v-D)^+, \tag{10.2.27}$$

and

$$\begin{aligned} l(x,\hat{v}(x)) &= p\bar{D} + (h-p)x + \mathbb{1}_{x\leq s}(K + (c-p)(S-x)) \\ &\quad + p[\mathbb{1}_{x\leq s}E(S-D)^+ + \mathbb{1}_{x>s}E(x-D)^+], \end{aligned} \tag{10.2.28}$$

and (10.2.23) reads

$$u(x) = l(x,\hat{v}(x)) - \rho + Eu(x + \hat{v}(x) - D). \tag{10.2.29}$$

Let us denote by $m_{\hat{v}(.)}(dx)$ the invariant measure $m_{s,S}(dx)$, for consistency of notation. It follows from (10.2.29) that

$$\rho = \int l(x,\hat{v}(x)) m_{\hat{v}(.)}(dx). \tag{10.2.30}$$

Note that the state space of $m_{\hat{v}(.)}(dx)$ is $[0, S]$ but can be taken as $[0,\infty)$ to work with a fixed state space. Now define the set $\mathcal{U}_b$ of feedbacks such that

$$\mathcal{U}_b = \{v(.) \geq 0 | \exists M \text{ such that } x + v(x) \leq \max(x, M)\}. \tag{10.2.31}$$

If $v(.) \in \mathcal{U}_b$, necessarily $v(x) = 0$, if $x \geq M$. The Markov chain, controlled with the feedback $v(.)$ is ergodic with state space $[0, M]$. We denote the corresponding invariant measure by $m_{v(.)}(dx)$. Note that a feedback defined by an s, S policy belongs to $\mathcal{U}_b$. We then have

Theorem 10.3. *We make the assumptions of Theorem 10.2. We have*

$$\rho = \inf_{v(.)\in\mathcal{U}_b} \int l(x,v(x)) m_{v(.)}(dx), \tag{10.2.32}$$

and there exists an optimal feedback, defined by an s, S policy. This s, S policy has been defined in Theorem 10.2.

PROOF. Consider a feedback $v(.)$ in $\mathcal{U}_b$. From (10.2.23) we can state

$$u(x) + \rho \leq l(x,v(x)) + Eu((x + v(x) - D)^+). \tag{10.2.33}$$

Integrating with respect to the invariant measure $m_{v(.)}(dx)$ yields

$$\rho \leq \int l(x,v(x)) m_{v(.)}(dx).$$

Taking account of (10.2.30) the result follows immediately. □

Remark 10.1. Let $J_{\alpha,x}(v(.))$ be the cost functional with discount α for a controlled Markov chain, with control defined by a feedback $v(.)$ belonging to $\mathcal{U}_b$. From Ergodic Theory we can claim that

$$(1-\alpha)J_{\alpha,x}(v(.)) \to \int l(\xi, v(\xi))m_{v(.)}(d\xi). \tag{10.2.34}$$

Therefore we have also

$$\rho = \inf_{v(.)\in\mathcal{U}_b} \lim_{\alpha\to 1}(1-\alpha)J_{\alpha,x}(v(.)). \tag{10.2.35}$$

Let us also give the interpretation of the function $u(x)$. Let us pick a feedback in $\mathcal{U}_b$. Let us denote by y_n, v_n the corresponding trajectory, with

$$y_{n+1} = (y_n + v_n - D_n)^+ \quad v_n = v(y_n); \quad y_1 = x.$$

Lemma 10.2. *We have the property*

$$Eu(y_n) \to \int u(x)m_{v(.)}(dx). \tag{10.2.36}$$

PROOF. The integral is well defined because the function $u(x)$ is continuous and the range of $m_{v(.)}(dx)$ is $[0, M]$. In fact (10.2.36) will hold for any continuous function. By taking positive and negative parts of $u(x)$ we may as well assume that $u(x) \geq 0$. Consider the sequence $u_\epsilon(x) = \dfrac{u(x)}{1+\epsilon u(x)}$. We note that $0 \leq y_n \leq \max(x, M)$. So $Eu(y_n) < \infty$ and we can write

$$Eu(y_n) = Eu_\epsilon(y_n) + \epsilon E\frac{u^2(y_n)}{1+\epsilon u(y_n)}, \tag{10.2.37}$$

and $\dfrac{u^2(y_n)}{1+\epsilon u(y_n)} \leq C = \max_{\xi\in[0,\max(x,M)]} u^2(\xi)$. If the initial inventory $x > M$, we note that

$$\{y_{n+1} > M\} = \{D_1 + \cdots + D_n < x - M\}.$$

Therefore

$$P(\{y_{n+1} > M\}) = F^{(n)}(x-M) \leq F^n(x-M).$$

Now

$$\limsup_{n\to\infty} Eu_\epsilon(y_n) = \limsup_{n\to\infty} Eu_\epsilon(y_{n+n_0}) \leq \frac{1}{\epsilon}P(y_{n_0} > M) + \limsup_{n\to\infty} E\mathbb{1}_{y_{n_0}\leq M} u_\epsilon(y_{n+n_0}).$$

From Ergodic Theory, using the fact that $u_\epsilon(x)$ is bounded (by $\frac{1}{\epsilon}$) we have

$$E\mathbb{1}_{y_{n_0}\leq M} u_\epsilon(y_{n+n_0}) \to \int u_\epsilon(x)m_{v(.)}(dx), \quad \text{as } n\to\infty.$$

Therefore

$$\limsup_{n\to\infty} Eu_\epsilon(y_n) \leq \int u_\epsilon(x)m_{v(.)}(dx) + \frac{1}{\epsilon}P(y_{n_0} > M).$$

Letting $n_0 \to \infty$ we obtain

$$\limsup_{n\to\infty} Eu_\epsilon(y_n) \leq \int u_\epsilon(x)m_{v(.)}(dx).$$

On the other hand, we have also

$$\liminf_{n\to\infty}\ Eu_\epsilon(y_n) = \liminf_{n\to\infty} Eu_\epsilon(y_{n+n_0}) \geq \liminf_{n\to\infty}\ E\mathbb{1}_{y_{n_0}\leq M} u_\epsilon(y_{n+n_0}).$$

Therefore

$$\liminf_{n\to\infty} \ Eu_\epsilon(y_n) \geq \int u_\epsilon(x)m_{v(.)}(dx).$$

Hence

$$Eu_\epsilon(y_n) \to \int u_\epsilon(x)m_{v(.)}(dx),$$

as $n \to \infty$. Letting now $\epsilon \to 0$ and using Fatou's Lemma we deduce

$$\int u_\epsilon(x)m_{v(.)}(dx) \to \int u(x)m_{v(.)}(dx).$$

On the other hand, from (10.2.37), we have

$$|Eu(y_n) - Eu_\epsilon(y_n)| \leq C\epsilon,$$

hence

$$\limsup_{n\to\infty} Eu(y_n) \leq \int u_\epsilon(x)m_{v(.)}(dx) + C\epsilon.$$

Letting $\epsilon \to 0$, we get

$$\limsup_{n\to\infty} Eu(y_n) \leq \int u(x)m_{v(.)}(dx).$$

Now,

$$\liminf_{n\to\infty} Eu(y_n) \geq \liminf_{n\to\infty} Eu_\epsilon(y_n) = \int u_\epsilon(x)m_{v(.)}(dx).$$

Letting $\epsilon \to 0$, and using Lebesgue's Theorem we obtain

$$\liminf_{n\to\infty} Eu(y_n) \geq \int u(x)m_{v(.)}(dx),$$

and the result follows. □

Let us consider a feedback $v(.) \in \mathcal{U}_b$ and let us consider y_n, v_n the corresponding trajectory

Define

(10.2.38) $$J_x^n(v(.)) = \sum_{j=1}^{n} E(l(y_j, v_j) - n\rho.$$

We then assert

Theorem 10.4. *We make the assumptions of Theorem 10.2. Then $J_x^n(\hat{v}(.))$ has a limit as $n \to \infty$ and*

(10.2.39) $$u(x) = \lim_{n\to\infty} J_x^n(\hat{v}(.)) + \int u(x)m_{\hat{v}(.)}(dx).$$

Moreover, for $v(.) \in \mathcal{U}_b$, $J_x^n(v(.))$ is bounded below, and on has

(10.2.40) $$u(x) = \inf_{v(.)\in\mathcal{U}_b} \left[\liminf_{n\to\infty} J_x^n(v(.)) + \int u(x)m_{v(.)}(dx)\right].$$

Proof. From equation (10.2.29) we check easily

$$u(x) = J_x^n(\hat{v}(.)) + Eu(\hat{y}_{n+1}),$$

and from (10.2.36) we deduce easily the property (10.2.39). Similarly, considering any feedback, which belongs to $\mathcal{U}_b$, we can write using (10.2.31)

$$u(x) \leq J_x^n(v(.)) + Eu(y_{n+1}).$$

Using again (10.2.36), we obtain that

$$u(x) \leq \liminf_{n\to\infty} J_x^n(v(.)) + \int u(x) m_{v(.)}(dx),$$

and the property (10.2.40) follows. □

We see that the function $u(.)$ enters in the definition of the objective function.

10.2.4. PARTICULAR CASE. We consider the particular case of an exponential distribution $f(x) = \beta \exp -\beta x$. The solution $\Gamma(x)$ of the renewal equation (10.2.13) is easily seen to be

$$\Gamma(x) = 1 + \beta x. \tag{10.2.41}$$

Next we have

$$\mu(x) = h - (p - c + h) \exp -\beta x, \tag{10.2.42}$$

and

$$g(x) = hx + \frac{p}{\beta} - \frac{p-c+h}{\beta}(1 - \exp -\beta x). \tag{10.2.43}$$

We then compute, for $x > s$

$$\begin{aligned} H(x) &= \int_s^x \Gamma(x-\xi)\mu(\xi)d\xi \\ &= h(x-s) + \frac{\beta h}{2}(x-s)^2 - (p-c+h)(x-s)\exp -\beta s \end{aligned} \tag{10.2.44}$$

The number $\bar{s}$ satisfies $\mu(\bar{s})$=0, hence

$$\exp -\beta \bar{s} = \frac{h}{p-c+h}. \tag{10.2.45}$$

We thus consider $0 \leq s \leq \bar{s}$. We define $S(s)$ by $H'(S) = 0$, hence

$$h + \beta h(S - s) - (p - c + h)\exp -\beta s = 0. \tag{10.2.46}$$

This equation has a unique solution $S = S(s)$, for $s \leq \bar{s}$, and one easily checks that $S \geq \bar{s}$. We next define the value of s. We need a condition, see (10.2.19). Let $S_0 = S(0)$. From (10.2.46) applied with $s = 0$, we obtain $S_0 = \dfrac{p-c}{\beta h}$ and next, from formula (10.2.44), with $s = 0$, we see that $H_0(S_0) = -\dfrac{(p-c)^2}{2\beta h}$ and the condition on K is

$$K < \frac{(p-c)^2}{2\beta h}. \tag{10.2.47}$$

To define s, we must write the equation $H_s(S(s)) = -K$. Using formula (10.2.44) and (10.2.46) we check that

$$H_s(S(s)) = -\frac{[(p-c+h)\exp -\beta s - h]^2}{2\beta h},$$

and we obtain easily the value of s, by

$$\exp -\beta s = \frac{h + \sqrt{2\beta h K}}{p-c+h}. \tag{10.2.48}$$

Thanks the condition (10.2.47) the right hand side of (10.2.48) is less than one, so $s < \bar{s}$ is well defined. Using (10.2.48) in (10.2.46), we can check the formula

$$S - s = \sqrt{\frac{2K}{\beta h}} = \sqrt{\frac{2K\bar{D}}{h}}. \tag{10.2.49}$$

Remark 10.2. If we compare with the deterministic case (10.1.3) where we had an s, S policy with $s = D$ and $S = D(\hat{k}+1)$, so $S - s = D\hat{k}$. The integer $\hat{k}$ minimizes $\frac{K}{k+1} + hD\frac{k}{2}$. Assimilating k and $k+1$ to the same real number, we obtain $\hat{k} = \sqrt{\frac{2K}{hD}}$ and thus $S - s = \sqrt{\frac{2KD}{h}}$, which is exactly (10.2.49) with of course $\bar{D} = D$. So the ordering quantity is the same as in the deterministic equivalent, but the ordering point s is not $\bar{D}$. It takes into account the randomness, and also the penalty when some demand is not satisfied. This cannot occur in the deterministic case.

We can next obtain the number $\rho = g(s)$. Using (10.2.48) we obtain

$$\rho = hs + \frac{c}{\beta} + \frac{\sqrt{2\beta hK}}{\beta},$$

and thus also

$$\rho = \frac{c}{\beta} + hS, \tag{10.2.50}$$

which is also equivalent to the deterministic case, see(10.1.4). Let us recover this formula, by the probabilistic interpretation, Theorem 10.3. We will not take any feedback in $\mathcal{U}_b$, but only those defined by an s, S policy. So consider a feedback defined by an s, S policy. We call it $v_{s,S}$. We consider the invariant measure of the corresponding controlled Markov chain. We call it $m_{s,S}(dx)$, see (9.2.82)

$$m_{s,S}(dx) = \frac{\exp -\beta s}{1 + \beta(S-s)}\delta(x) + \frac{\beta \exp -\beta(x-s)^-}{1+\beta(S-s)}dx. \tag{10.2.51}$$

Let us call $l_{s,S}(x) = l(x, v_{s,S}(x))$. We check easily the formula

$$l_{s,S}(x) = (K - (p-c)(S-x))1\!\!1_{x \le s} + p\bar{D} + (h-p)x \tag{10.2.52}$$

$$+p1\!\!1_{x>s}E(x-D)^+ + p1\!\!1_{x\le s}E(S-D)^+. \tag{10.2.53}$$

This formula is general and is not limited to the particular case of exponential distribution. We then compute the ergodic cost

$$J_{s,S} = \int_0^S l_{s,S}(x) m_{s,S}(dx).$$

Making use of the property of the invariant measure, when integrating the last part of relation (10.2.52), we obtain

$$J_{s,S} = p\bar{D} + (K - (p-c)S)\int_0^s m_{s,S}(dx) \tag{10.2.54}$$

$$+(p-c)\int_0^s x m_{s,S}(dx) + h\int_0^S x m_{s,S}(dx).$$

We now use the particular case, in which the invariant measure is given by (10.2.51) and $\bar{D} = \frac{1}{\beta}$. After some lengthy but easy calculations we get the formula

$$J_{s,S} = \frac{c - \frac{h}{2}}{\beta} + \frac{K + \frac{p-c+h}{\beta}\exp -\beta s - \frac{h}{2\beta}}{1+\beta(S-s)} + h(\frac{1}{2}(S-s)+s). \tag{10.2.55}$$

We can try now to minimize this expression in the two arguments $s, S-s$. We first consider s fixed and minimize in $S-s$. We obtain the condition

$$\frac{K + \frac{p-c+h}{\beta}\exp -\beta s - \frac{h}{2\beta}}{(1+\beta(S-s))^2} = \frac{h}{2\beta}, \tag{10.2.56}$$

which defines $S(s)$. We can then compute $J_s = J_{s,S(s)}$. We obtain the simple expression

$$J_s = \frac{c}{\beta} + hS(s). \tag{10.2.57}$$

Since we want to minimize J_s, we find s such that $S'(s) = 0$. Now writing (10.2.56) as

$$(1+\beta(S-s))^2 = \frac{2\beta}{h}K + 2\frac{p-c+h}{h}\exp -\beta s - 1.$$

We differentiate this expression in s, and we take into account that $S'(s) = 0$. We check easily that this implies

$$1+\beta(S-s) = \frac{p-c+h}{h}\exp -\beta s,$$

which is exactly (10.2.46). Using this relation in (10.2.56) and rearranging we obtain

$$K = \frac{h\beta}{2}(S-s)^2,$$

which is (10.2.49). So the number ρ is the minimum of the ergodic cost, over all possible s, S policies.

10.3. ERGODIC INVENTORY CONTROL WITH FIXED COST AND BACKLOG

10.3.1. STATEMENT OF THE PROBLEM. We consider the Bellman equation

$$u_\alpha(x) = -cx + hx^+ + px^- + \inf_{\eta \geq x}\{K 1\!\!1_{\eta > x} + c\eta + \alpha E u_\alpha(\eta - D)\}, \tag{10.3.1}$$

studied in section 9.3. Define

$$g_\alpha(x) = cx(1-\alpha) + \alpha c\bar{D} + \alpha hE(x-D)^+ + \alpha pE(x-D)^-, \tag{10.3.2}$$

and

$$\mu_\alpha(x) = g'_\alpha(x) = c(1-\alpha) + \alpha h - \alpha(h+p)\bar{F}(x). \tag{10.3.3}$$

Then

$$H_{\alpha,s}(x) = \int_s^x \Gamma_\alpha(x-\xi)\mu_\alpha(\xi)d\xi, \tag{10.3.4}$$

where $\Gamma_\alpha(x)$ is the solution of the renewal equation

$$\Gamma_\alpha(x) = 1 + \alpha \int_0^x f(x-\xi)\Gamma_\alpha(\xi)\,d\xi. \tag{10.3.5}$$

We define the function $S_\alpha(s)$ such that

$$\inf_{\eta \geq s} H_{\alpha,s}(\eta) = H_{\alpha,s}(S_\alpha(s)), \tag{10.3.6}$$

and the pair s_α, S_α is defined by the relations

$$\inf_{\eta \geq s_\alpha} H_{\alpha,s_\alpha}(\eta) + K = 0, \quad S_\alpha = S_\alpha(s_\alpha). \tag{10.3.7}$$

We know that

$$s_\alpha \leq \bar{s}_\alpha, \tag{10.3.8}$$

with

$$c(1-\alpha) + \alpha h - \alpha(p+h)\bar{F}(\bar{s}_\alpha) = 0. \tag{10.3.9}$$

Then

$$u_\alpha(x) = H_{\alpha,s_\alpha}(x) - cx + hx^+ + px^- + \frac{g_\alpha(s_\alpha)}{1-\alpha}. \tag{10.3.10}$$

We want to study the behavior of $u_\alpha(x)$ as $\alpha \uparrow 1$.

10.3.2. CONVERGENCE.

We shall make the assumption

$$f(x) \text{ is continuous}, \quad f(x) > 0, \forall x. \tag{10.3.11}$$

We recall that $\bar{D} = \int_0^{+\infty} xf(x)dx < \infty$. Also $F(x) < 1, \forall x$.

The function $\Gamma_\alpha(x)$ converges to $\Gamma(x)$ where

$$\Gamma(x) = 1 + \int_0^x f(x-\xi)\Gamma(\xi)d\xi. \tag{10.3.12}$$

We have next

$$\mu_\alpha(x) \to \mu(x) = h - (p+h)\bar{F}(x). \tag{10.3.13}$$

Similarly

$$g_\alpha(x) \quad \to \quad g(x) = c\bar{D} + hE(x-D)^+ + pE(x-D)^-. \tag{10.3.14}$$

The function

$$H_{\alpha,s}(x) \to H_s(x) = \int_s^x \Gamma(x-\xi)\mu(\xi)d\xi. \tag{10.3.15}$$

We can define $\bar{s} > 0$ such that $\mu(\bar{s}) = 0$ and $\mu(x) < 0, \forall x < \bar{s}$ and $\mu(x) > 0, \forall x > \bar{s}$. Since Γ is an increasing function, the function $H_s(x)$ is increasing in x, whenever $s \geq \bar{s}$. Therefore we can assert

$$S(s) = s, \ \inf_{\eta \geq s} H_s(\eta) = 0, \text{ if } s \geq \bar{s}.$$

So we limit the set of possible s to satisfy $s < \bar{s}$.

Lemma 10.3. *For $s < \bar{s}$, the infimum of $H_s(x)$ for $x \geq s$ is attained in points strictly larger that $\bar{s}$. Moreover* $\inf_{x \geq s} H_s(x) < 0$.

PROOF. We cannot argue as in Lemma 10.1. The reason is that we do not have anymore the a priori bound provided by $S_\alpha(0)$. The equivalent would be $S_\alpha(-\infty)$. However, from formula (9.3.24) we see that $S_\alpha(-\infty)$ is not bounded as $\alpha \uparrow 1$. We shall instead rely on the estimates

$$1 + B_1 x \le \Gamma(x) \le 1 + B_2 x,\ B_2 = \sup_{x \ge 0} \frac{F(x)}{\int_0^x \bar{F}(\xi)d\xi},\ B_1 = \inf_{x \ge 0} \frac{F(x)}{\int_0^x \bar{F}(\xi)d\xi}. \tag{10.3.16}$$

Note that the numbers B_1, B_2 are well defined. Indeed the function $G(x) = \dfrac{F(x)}{\int_0^x \bar{F}(\xi)d\xi}$ converges to $f(0) > 0$ as $x \to 0$ and to $\dfrac{1}{\bar{D}}$ as $x \to \infty$. Moreover it is continuous. Call

$$z(x) = \Gamma(x) - 1 - Bx,$$

then z is the solution of

$$z(x) = F(x) - B\int_0^x \bar{F}(\xi)d\xi + \int_0^x z(x-\xi)f(\xi)d\xi.$$

If $F(x) - B\int_0^x \bar{F}(\xi)d\xi \le 0$, the function $z(x)$ is also negative. Similarly, if $F(x) - B\int_0^x \bar{F}(\xi)d\xi \ge 0$, it is positive. The estimates (10.3.16) follow easily. We next express

$$\begin{aligned} H_s(x) &= \int_s^x \Gamma(x-\xi)\mu(\xi)d\xi \\ &= h\int_s^x \Gamma(x-\xi)d\xi - (p+h)\int_s^x \Gamma(x-\xi)\bar{F}(\xi)d\xi. \end{aligned}$$

If $s > 0$, we get

$$H_s(x) \ge h\int_s^x \Gamma(x-\xi)d\xi - (p+h)\int_0^x \Gamma(x-\xi)\bar{F}(\xi)d\xi.$$

However

$$\int_0^x \Gamma(x-\xi)\bar{F}(\xi)d\xi = x, \tag{10.3.17}$$

which can be seen simply by differentiating the left hand side and showing that the derivative is 1. So, we have, from (10.3.16) and (10.3.17),

$$\begin{aligned} H_s(x) &\ge h(x-s) + \frac{hB_1}{2}(x-s)^2 - (p+h)x \\ &= \frac{hB_1}{2}(x-s)^2 - px - hs, \quad x > s > 0 \end{aligned} \tag{10.3.18}$$

Now if $s < 0$, we have

$$H_s(x) = -p\int_s^0 \Gamma(x-\xi)d\xi + h\int_0^x \Gamma(x-\xi)d\xi - (p+h)x,$$

and using (10.3.16) we get easily

$$H_s(x) \ge ps + p\frac{B_2}{2}(-s^2 + 2sx) - px + h\frac{B_1}{2}x^2, s < 0 < x. \tag{10.3.19}$$

Therefore, in both cases, $H_s(x) \to \infty$ as $x \uparrow \infty$. Also $H_s'(s+) = \mu(s) < 0$. Hence the function $H_s(x)$ attains a negative minimum on (s, ∞). We call $S(s)$ the smallest minimum if there are many. $\square$

We have

$$H'_s(S(s)) = 0.$$

We have next

$$\frac{d}{ds}H_s(S(s)) = \frac{\partial}{\partial s}H_s(x)|_{S(s)} = -\Gamma(S(s)-s)\mu(s) > 0.$$

Moreover $S(s) > \bar{s} > 0$. For $s < 0$, $x > 0$ we have

$$H_s(x) = -p\int_s^0 \Gamma(x-\xi)d\xi + h\int_0^x \Gamma(\xi)d\xi - (p+h)x,\ \forall s < 0 < x. \tag{10.3.20}$$

It follows that

$$H'_s(x) = (p+h)\Gamma(x) - p\Gamma(x-s) - (p+h),$$

therefore $S(s)$ is solution of the equation

$$(p+h)\Gamma(S) - p\Gamma(S-s) - (p+h) = 0. \tag{10.3.21}$$

It follows that as $s \to -\infty$, $S(s) \to +\infty$. Otherwise, there will be a contradiction with (10.3.21). Furthermore, for $s < 0$ one has

$$H_s(S(s)) \le H_s(0) = -p\int_0^{-s}\Gamma(\xi)d\xi \le ps,$$

from which it follows that $H_s(S(s)) \to -\infty$ as $s \to -\infty$.

Hence the function $\inf_{\eta\ge s} H_s(\eta)$ increases from $-\infty$ to 0, as s increases from $-\infty$ to $\bar{s}$. Therefore there exists a unique s for which it is equal to $-K$. We can find the pair s, S by solving the algebraic system

$$\begin{aligned} &\Gamma(S-s)\mu(s) + \int_s^S \Gamma(S-\xi)\mu'(\xi)d\xi = 0 \\ &\int_s^S \Gamma(S-\xi)\mu(\xi)d\xi = -K \end{aligned} \tag{10.3.22}$$

Theorem 10.5. *We assume $h < c$ and (10.3.11), then we have*

$$s_\alpha \to s, \quad S_\alpha \to S, \tag{10.3.23}$$

where the pair s, S is solution of (10.3.22). Let $H(x) = H_s(x)$. Then

$$u_\alpha(x) - \frac{g_\alpha(s_\alpha)}{1-\alpha} \to u(x) = H(x) - cx + hx^+ + px^-. \tag{10.3.24}$$

Set $\rho = g(s)$, then $\rho > 0$ and one has

$$u(x) + \rho = -cx + hx^+ + px^- + \inf_{\eta\ge x}\{K 1\!\!1_{\eta>x} + c\eta + Eu(\eta - D)\}. \tag{10.3.25}$$

PROOF. Let us check that s_α is bounded. Since $s_\alpha < \bar{s}_\alpha \to \bar{s}$, it is sufficient to assume $s_\alpha < 0$. We have $S_\alpha > 0$. We can write

$$\begin{aligned} -K &= H_{\alpha,s_\alpha}(S_\alpha) \le H_{\alpha,s_\alpha}(0) \\ &= (c(1-\alpha)-\alpha p)\int_{s_\alpha}^0 \Gamma_\alpha(-\xi)d\xi \le -s_\alpha(c(1-\alpha)-\alpha p) \end{aligned}$$

which implies

$$s_\alpha^- \le \frac{K}{\alpha p - c(1-\alpha)},$$

and thus s_α remains bounded. But then, using estimates (10.3.18), (10.3.19) we get immediately that S_α remains also bounded. So we can extract a subsequence which converges to s^*, S^*. From the expression of $H_{\alpha,s}(x)$, we easily deduce that $H_{\alpha,s_\alpha}(S_\alpha) = -K \to H_{s^*}(S^*)$. From this one obtains that $s^* = s$, $S^* = S$, solutions of equations (10.3.22). The value s is uniquely defined, and since S_α is the smallest infimum, we obtain also that S is the smallest minimum.

Note that

$$g(x) = c\bar{D} + hE(x-D)^+ + pE(x-D)^-,$$

so it is a positive function. Hence $\rho > 0$. Clearly

$$(1-\alpha)u_\alpha(x) - g_\alpha(s_\alpha) \to 0, \text{ as } \alpha \uparrow 1,$$

hence

$$(1-\alpha)u_\alpha(x) \to \rho = g(s).$$

We recall that we have also

$$H(x) + \rho = g(x) + EH(x-D), \; \forall x \geq s$$

$$H(x) = 0, \forall x \leq s$$

where $H(x) = H_s(x)$, defined by (10.3.15), for the specific s, limit of s_α.

We can check that these conditions can be summarized into

$$H(x) + \rho = + \inf_{\eta \geq x} [K1\!\!1_{\eta > x} + g(\eta) + EH(\eta - D)]. \tag{10.3.26}$$

If (10.3.26) is satisfied, then the equation for u (10.3.25) follows immediately, using the definition of $g(x)$.

The proof of (10.3.26) is identical to that of (10.2.25) in Theorem 10.2.

This completes the proof of the Theorem. □

10.3.3. PROBABILISTIC INTERPRETATION. We give now the interpretation of ρ and of the solution $u(x)$ of (10.3.25). We first associate to an s, S policy an invariant measure $m_{s,S}(dx)$. Moreover the infimum in equations (10.3.25), (10.3.26) is attained by a feedback $\hat{v}(x)$ associated to an s, S policy. We still denote it s, S to save notation.

We recall that

$$l(x,v) = K1\!\!1_{v>0} + cv + hx^+ + px^-, \tag{10.3.27}$$

and

$$l(x, \hat{v}(x)) = 1\!\!1_{x \leq s}(K + c(S-x)) + hx^+ + px^-, \tag{10.3.28}$$

and (10.3.25) reads

$$u(x) = l(x, \hat{v}(x)) - \rho + Eu(x + \hat{v}(x) - D). \tag{10.3.29}$$

Let us denote by $m_{\hat{v}(.)}(dx)$ the invariant measure $m_{s,S}(dx)$, for consistency of notation. It follows from (10.3.29) that

$$\rho = \int l(x, \hat{v}(x)) m_{\hat{v}(.)}(dx). \tag{10.3.30}$$

Note that the state space of $m_{\hat{v}(.)}(dx)$ is $(-\infty, S]$ but can be taken as $(-\infty, \infty)$ to work with a fixed state space (independent of S). We next define the set $\mathcal{U}_b$ of feedbacks such that

$$\mathcal{U}_b = \{v(.) \geq 0 | \exists M \text{ such that } -M \leq x + v(x) \leq \max(x, M)\}. \tag{10.3.31}$$

If $v(.) \in \mathcal{U}_b$, necessarily $v(x) = 0$, if $x \geq M$. The Markov chain, controlled with the feedback $v(.)$ is ergodic with state space $(-\infty, M]$. Indeed, the transition probability is

$$\pi(x; d\eta) = \mathbb{1}_{\eta < x+v(x)} f(x + v(x) - \eta) d\eta.$$

If we take $X_0 = [-M-2, -M-1]$, then for $\eta \in X_0$, we have

$$\mathbb{1}_{\eta < x+v(x)} f(x + v(x) - \eta) \geq \min_{y \in [1, 2M+2]} f(y) > 0.$$

We denote the corresponding invariant measure by $m_{v(.)}(dx)$. Note that a feedback defined by an s, S policy belongs to $\mathcal{U}_b$. We then have

Theorem 10.6. *We make the assumptions of Theorem 10.5. We have*

$$\rho = \inf_{v(.) \in \mathcal{U}_b} \int l(x, v(x)) m_{v(.)}(dx), \tag{10.3.32}$$

and there exists an optimal feedback, defined by an s, S policy. This s, S policy has been defined in Theorem 10.5.

PROOF. Consider a feedback $v(.)$ in $\mathcal{U}_b$. From (10.3.25) we can state

$$u(x) + \rho \leq l(x, v(x)) + Eu(x + v(x) - D). \tag{10.3.33}$$

Integrating with respect to the invariant measure $m_{v(.)}(dx)$ yields

$$\rho \leq \int l(x, v(x)) m_{v(.)}(dx).$$

The integral is finite. Indeed, one has

$$\int |x| m_{v(.)}(dx) = \int E|x + v(x) - D| m_{v(.)}(dx) \leq M + \bar{D}.$$

Since

$$l(x, v(x)) = K\mathbb{1}_{v(x)>0} + c(x + v(x)) - cx + hx^+ + px^-,$$

it has a finite integral with respect to $m_{v(.)}(dx)$. Recalling also that

$$u(x) = H(x) - cx + hx^+ + px^-,$$

and noting that

$$\int |H(x)| m_{v(.)}(dx) = \int_s^M |H(x)| m_{v(.)}(dx) < \infty,$$

we see also that $|u(.)|$ is integrable with respect to $m_{v(.)}(dx)$.

Taking account of (10.3.30) the result follows immediately. □

Remark 10.3. Let $J_{\alpha,x}(v(.))$ be the cost functional with discount α for a controlled Markov chain, with control defined by a feedback $v(.)$ belonging to $\mathcal{U}_b$. From Ergodic Theory we can claim that

$$(1 - \alpha) J_{\alpha,x}(v(.)) \to \int l(\xi, v(\xi)) m_{v(.)}(d\xi). \tag{10.3.34}$$

Therefore we have also

$$\rho = \inf_{v(.) \in \mathcal{U}_b} \lim_{\alpha \to 1} (1 - \alpha) J_{\alpha,x}(v(.)). \tag{10.3.35}$$

Let us also give the interpretation of the function $u(x)$. Let us pick a feedback in $\mathcal{U}_b$. Let us denote by y_n, v_n the corresponding trajectory, with

$$y_{n+1} = y_n + v_n - D_n \quad v_n = v(y_n); \quad y_1 = x$$

Lemma 10.4. *We have the property*

$$Eu(y_n) \to \int u(x)m_{v(.)}(dx). \tag{10.3.36}$$

PROOF. The integral is well defined, as it has been seen in the proof of Theorem 10.6. However, we cannot proceed as in the proof of Lemma 10.2, because y_n does not remain in a compact interval. However we have

$$E\varphi(y_n) \to \int \varphi(x)m_{v(.)}(dx), \tag{10.3.37}$$

for any function $\varphi(x)$ continuous and bounded on the state space $(-\infty, M]$. This is done as in Lemma 10.2, using ergodic theory and the procedure to address the situation of an initial value $x > M$. Recalling that

$$u(x) = H(x) - cx + hx^+ + px^-,$$

and noting that $H(x)$ is continuous and bounded on $(-\infty, M]$, it remains to prove the property (10.3.37) with $\varphi(x) = x^+$ or x^-. For x^+ it is immediate, since it is continuous and bounded on $(-\infty, M]$. For x^-, one notices that,

$$Ey_{n+1}^- = EG(y_n),$$

where

$$G(x) = E(x + v(x) - D)^-,$$

and one notices that

$$G(x) \leq M + \bar{D},$$

therefore it is continuous and bounded. The proof has been completed. □

Let us consider a feedback $v(.) \in \mathcal{U}_b$ and let us consider y_n, v_n the corresponding trajectory

Define

$$J_x^n(v(.)) = \sum_{j=1}^{n} E(l(y_j, v_j) - n\rho. \tag{10.3.38}$$

We then assert

Theorem 10.7. *We make the assumptions of Theorem 10.5. Then $J_x^n(\hat{v}(.))$ has a limit as $n \to \infty$ and*

$$u(x) = \lim_{n\to\infty} J_x^n(\hat{v}(.)) + \int u(x)m_{\hat{v}(.)}(dx). \tag{10.3.39}$$

Moreover, for $v(.) \in \mathcal{U}_b$, $J_x^n(v(.))$ is bounded below, and on has

$$u(x) = \inf_{v(.)\in\mathcal{U}_b} \left[\liminf_{n\to\infty} J_x^n(v(.)) + \int u(x)m_{v(.)}(dx)\right]. \tag{10.3.40}$$

PROOF. The proof is the same as that of Theorem 10.4 □

10.3.4. PARTICULAR CASE. We consider the particular case of an exponential distribution $f(x) = \beta \exp -\beta x$. We recall that

$$\Gamma(x) = 1 + \beta x.$$

Next we have

$$\mu(x) = h - (p+h) \exp -\beta x^+, \tag{10.3.41}$$

and

$$g(x) = hx^+ + px^- + (c+p)\bar{D} - \frac{p+h}{\beta}(1 - \exp -\beta x^+). \tag{10.3.42}$$

We next compute the function $H(x)$, for $x > s$. We use the formulas

$$H'(x) = \begin{cases} -p\Gamma(x-s), & \text{if } x < 0 \\ -p\Gamma(x-s) + (p+h)(\Gamma(x) - 1), & \text{if } s < 0 < x \\ h\Gamma(x-s) - (p+h)\bar{F}(x) - (p+h)\int_s^x \Gamma'(x-\xi)\bar{F}(\xi)d\xi & \text{if } 0 < s < x \end{cases}$$

hence

$$H'(x) = \begin{cases} -p(1 + \beta(x-s)), & \text{if } x < 0 \\ -p(1 + \beta(x-s)) + (p+h)\beta x, & \text{if } s < 0 < x \\ h(1 + \beta(x-s)) - (p+h)\exp -\beta s & \text{if } 0 < s < x \end{cases}$$

We then obtain, for $x > s$

$$H(x) = -p\left(x - s + \frac{\beta}{2}(x-s)^2\right) + (h+p)\left[(x^+ - s^+)(1 - \exp -\beta s^+) + \frac{\beta}{2}(x^+ - s^+)^2\right]. \tag{10.3.43}$$

The number $\bar{s}$ satisfies $\mu(\bar{s})$=0, hence

$$\exp -\beta\bar{s} = \frac{h}{p+h}. \tag{10.3.44}$$

We thus consider $s \leq \bar{s}$. We define $S(s)$ by $H'(S) = 0$, hence

$$S(s) = \frac{p\beta s^- - h(1 - \beta s^+) + (h+p)\exp -\beta s^+}{h\beta}. \tag{10.3.45}$$

We can see that $S(\bar{s}) = \bar{s}$, and $S(s)$ is decreasing for $s \leq \bar{s}$. Moreover $S(-\infty) = +\infty$.

We next define the value of s. We first obtain the formula

$$H_s(S(s)) = -\frac{1}{2h\beta}[(h+p)\exp -\beta s^+ - h]^2 - \frac{\beta p}{2h}(p+h)(s^-)^2 - \frac{p(p+h)}{h}s^-. \tag{10.3.46}$$

We can check directly on this formula that $H_s(S(s))$ increases from $-\infty$ to 0, as s grows from $-\infty$ to $\bar{s}$. So there exists a unique s such that

$$H_s(S(s)) = -K.$$

Exercise 10.1. Check that

$$K < \frac{p^2}{2h\beta} \Longrightarrow (p+h)\exp -\beta s = h + \sqrt{2h\beta K},\ s > 0, \tag{10.3.47}$$

and for $K > \dfrac{p^2}{2h\beta}$ then s is the negative root of

$$\frac{s^2\beta p(p+h)}{2h} - \frac{sp(p+h)}{h} + \frac{p^2}{2h\beta} - K = 0. \tag{10.3.48}$$

We can next obtain the number $\rho = g(s)$. We first have, from (10.3.46)

$$(h+p)\exp -\beta s^{+} = h + \sqrt{2h\beta}\sqrt{K - \frac{p(p+h)}{h}s^{-}\left(1+\beta\frac{s^{-}}{2}\right)},$$

and then using (10.3.42), we obtain

$$\rho = hs^{+} + ps^{-} + \frac{c}{\beta} + \frac{\sqrt{2h\beta}}{\beta}\sqrt{K - \frac{p(p+h)}{h}s^{-}\left(1+\beta\frac{s^{-}}{2}\right)}, \tag{10.3.49}$$

and using the formula for S, we get again (10.2.50), namely

$$\rho = \frac{c}{\beta} + hS. \tag{10.3.50}$$

Let us recover this formula, by the probabilistic interpretation, Theorem 10.6. We will consider only s, S policies.

So consider a feedback defined by an s, S policy. We call it $v_{s,S}$. We consider the invariant measure of the corresponding controlled Markov chain. We call it $m_{s,S}(dx)$, see (9.3.45)

$$m_{s,S}(dx) = \frac{\beta \exp -\beta(x-s)^{-}}{1+\beta(S-s)}dx. \tag{10.3.51}$$

Let us call $l_{s,S}(x) = l(x, v_{s,S}(x))$. We recall formula (10.3.28) and we compute the ergodic cost

$$J_{s,S} = \int_0^S l_{s,S}(x) m_{s,S}(dx).$$

We obtain

$$J_{s,S} = K\int_0^s m_{s,S}(dx) + c\int_0^s (S-x)m_{s,S}(dx) + h\int_0^S x m_{s,S}(dx) - p\int_{-\infty}^0 x m_{sS}(dx). \tag{10.3.52}$$

Using the invariant measure given by (10.3.51) we get the formula

$$J_{s,S} = \frac{c}{\beta} + \frac{K - \frac{h}{\beta} + \frac{h+p}{\beta}\exp -\beta s^{+} + h\beta\frac{S^2}{2} + hs^{+} + ps^{-} + \frac{p\beta}{2}(s^{-})^2 - \frac{h\beta}{2}(s^{+})^2}{1+\beta(S-s)}. \tag{10.3.53}$$

We can try now to minimize this expression in the two arguments $s, S-s$. We first consider s fixed and minimize in S. We obtain the condition

$$\frac{K - \frac{h}{\beta} + \frac{h+p}{\beta}\exp -\beta s^{+} + h\beta\frac{S^2}{2} + hs^{+} + ps^{-} + \frac{p\beta}{2}(s^{-})^2 - \frac{h\beta}{2}(s^{+})^2}{1+\beta(S-s)} = hS, \tag{10.3.54}$$

which defines $S(s)$. We can then compute $J_s = J_{s,S(s)}$. We obtain the simple expression

$$J_s = \frac{c}{\beta} + hS(s). \tag{10.3.55}$$

Since we want to minimize J_s, we find s such that $S'(s) = 0$. Now writing (10.3.54) as

(10.3.56)

$$K - \frac{h}{\beta} + \frac{h+p}{\beta} \exp -\beta s^+ - h\beta\frac{S^2}{2} + hs^+ + ps^- + \frac{p\beta}{2}(s^-)^2 - \frac{h\beta}{2}(s^+)^2 - hS + h\beta S s = 0.$$

We differentiate this expression in s, and we take into account that $S'(s) = 0$. We check easily that this implies

$$h\beta S = -h + \beta(hs^+ + ps^-) + (h+p)\exp -\beta s^+,$$

which is exactly (10.3.45). Using this relation in (10.3.56) and rearranging we obtain

$$K - \frac{1}{2h\beta}[(h+p)\exp -\beta s^+ - h]^2 - \frac{\beta p}{2h}(p+h)(s^-)^2 - \frac{p(p+h)}{h}s^- = 0,$$

which is $H_s(S(s)) = -K$. So the number ρ is the minimum of the ergodic cost, over all possible s, S policies.

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CHAPTER 11

DYNAMIC INVENTORY MODELS WITH EXTENSIONS

11.1. CAPACITATED INVENTORY MANAGEMENT

We consider the dynamic inventory model, without set-up cost, see Chapter 5. To fix the ideas we study only the case with backlog, see section 5.2. By capacitated inventory management, we mean that there is a limit in the size of the order. Therefore, if x denotes the stock before the order, and η the stock just after the order, then we have the constraints

$$x \leq \eta \leq x + Q,$$

where Q represents the bound on the size of the order. We adapt the Bellman equation (5.2.5), as follows

$$u(x) = \inf_{0 \leq v \leq Q} [l(x, v) + \alpha E u(x + v - D)], \tag{11.1.1}$$

with

$$l(x, v) = hx^+ + px^- + cv, \tag{11.1.2}$$

so

$$u(x) = hx^+ + px^- - cx + \inf_{x \leq \eta \leq x+Q} [c\eta + \alpha E u(\eta - D)]. \tag{11.1.3}$$

We first state

Theorem 11.1. *Under the assumption* (11.1.2), *the solution of* (11.1.1) *in the space* B_1 *is unique. Moreover* u *is continuous. There exists an optimal feedback* $\hat{v}(x, Q)$.

PROOF. The proof is similar to that of Theorem 5.5 and even simpler, since the control v is bounded. We omit details. □

We want to investigate the modifications with respect to the base stock policy. Let us denote by $u(x, Q)$ the solution of (11.1.1). We state

Theorem 11.2. *Assume,*

$$c(1 - \alpha) - p\alpha < 0, \quad f(x) \text{ continuous, bounded.} \tag{11.1.4}$$

The solution $u(x, Q)$ *of* (11.1.1) *is convex,* C^1 *in* x, *except in* 0, *where it has a left and right derivative and* $u(x, Q) \to +\infty$, *as* $|x| \to +\infty$. *Let*

$$G(x, Q) = cx + \alpha E u(x - D, Q). \tag{11.1.5}$$

Then $G(x,Q)$ is convex, C^1,$G(x,Q) \to +\infty$ as $|x| \to +\infty$. Let $S(Q)$ be the minimum of $G(x,Q)$, then the feedback $\hat{v}(x,Q)$ is given by the formula

$$\hat{v}(x,Q) = \begin{cases} Q, & \text{if } x < S(Q) - Q \\ S(Q) - x, & \text{if } S(Q) - Q < x < S(Q) \\ 0, & \text{if } x > S(Q) \end{cases} \tag{11.1.6}$$

$S(Q)$ is decreasing in Q and strictly positive. The derivative $u'_x(x,Q)$ increases in Q, except in 0. In 0, the property applies for the left and right derivatives.

Proof. We consider the monotone increasing process $u_n(x)$. We start with $u_0(x) = 0$, $u_1(x,Q) = hx^+ + px^-$ and

$$u_{n+1}(x,Q) = hx^+ + px^- - cx + \inf_{\{x \le \eta \le x+Q\}} [c\eta + \alpha E u_n(\eta - D, Q)]. \tag{11.1.7}$$

As we have seen in the case $Q = +\infty$, see Chapter 5, section 5.2, for $1 \le n \le n_0 + 1$ with

$$\frac{\alpha p}{1-\alpha}(1 - \alpha^{n_0+1}) > c \ge \frac{\alpha p}{1-\alpha}(1 - \alpha^{n_0}),$$

we have $u_n(x,Q) = u_n(x)$, with

$$u_{n+1}(x) = hx^+ + px^- + \alpha E u_n(x - D), 0 \le n \le n_0. \tag{11.1.8}$$

This is obvious, when $n_0 = 0$, which corresponds to $\alpha p > c$. So we can assume that $c \ge \alpha p$. We have $n_0 \ge 1$.

The functions $u_n(x)$ defined by the sequence (11.1.8) are convex and

$$u'_n(x) = -p\frac{1-\alpha^n}{1-\alpha},\ x < 0, 1 \le n \le n_0 + 1,$$

therefore

$$G_n(x) = cx + \alpha E u_n(x - D),$$

is also convex and monotone increasing, for $0 \le n \le n_0$. Note that $G_n(x)$ is C^1 since

$$G'_n(x) = c + \alpha \int_{-\infty}^{x} u'_n(z) f(x - z) dz,$$

and

$$G'_n(x) = c - \alpha p\frac{1-\alpha^n}{1-\alpha} \ge 0, \forall x \le 0,\ 0 \le n \le n_0$$

It follows that $u_n(x)$ satisfies (11.1.7), and therefore $u_n(x,Q) = u_n(x), \forall 1 \le n \le n_0 + 1$. Clearly $u_n(x) \to +\infty$, as $|x| \to +\infty$, and $u_n(x)$ is C^1 except in 0. One has

$$u'_n(0+) = h - \alpha p\frac{1-\alpha^{n-1}}{1-\alpha}, 1 \le n \le n_0 + 1$$

Consider now $u_{n_0+2}(x,Q)$. Note that

$$G'_{n_0+1}(x) = c - \alpha p\frac{1-\alpha^{n_0+1}}{1-\alpha},\ x \le 0$$

Therefore the function $G_{n_0+1}(x)$ is strictly decreasing for x negative and $\to +\infty$, as $x \to +\infty$. Hence it has a minimum.

Also the minimum is strictly positive. Taking the smallest minimum, we define it in a unique way and call it S_{n_0+1}. We then have

$$u_{n_0+2}(x,Q) = \begin{cases} hx^+ + px^- - cx + G_{n_0+1}(x+Q), & \text{if } x < S_{n_0+1} - Q \\ hx^+ + px^- - cx + G_{n_0+1}(S_{n_0+1}), & \text{if } S_{n_0+1} - Q < x < S_{n_0+1} \\ hx^+ + px^- - cx + G_{n_0+1}(x) & \text{if } x > S_{n_0+1} \end{cases}$$

We first see that this function is convex. This follows from the property

$$g_{n_0+1}(x) = \inf_{x \leq \eta \leq x+Q} G_{n_0+1}(\eta), \tag{11.1.9}$$

is convex, since $G_{n_0+1}(x)$ is convex. The function is clearly continuous. It is C^1 except in 0. Also $G_{n_0+1}(x) \to +\infty$, as $|x| \to +\infty$, and $u_{n_0+2}(x,Q) \to +\infty$, as $|x| \to +\infty$. We can also check that $u'_{n_0+2}(x,Q)$ is monotone increasing in Q, $\forall x \neq 0$, and the same is true for the left and right derivatives in 0. Then

$$G_{n_0+2}(x,Q) = cx + \alpha E u_{n_0+2}(x-D,Q),$$

is convex, C^1 and

$$\begin{aligned} G_{n_0+2}(x,Q) &\geq cx + \alpha[hE(x-D)^+ + pE(x-D)^- - c(x-\bar{D})] \\ &\geq [c(1-\alpha) - \alpha p]x \to +\infty, \text{as } x \to -\infty \end{aligned}$$

So $G_{n_0+2}(x,Q) \to +\infty$, as $|x| \to +\infty$. Also $G'_{n_0+2}(x,Q)$ is monotone increasing in Q. Moreover, for $x < 0$, from the expression above of $u_{n_0+2}(x,Q)$ we have $u'_{n_0+2}(x,Q) \leq -p - c$, and thus

$$G'_{n_0+2}(x,Q) \leq c(1-\alpha) - \alpha p < 0, \forall x \leq 0$$

Therefore $G_{n_0+2}(x,Q)$ attains its minimum in a point $S_{n_0+2}(Q) > 0$. By taking the smallest minimum, one can define in a unique way $S_{n_0+2}(Q)$. The function $S_{n_0+2}(Q)$ is monotone decreasing. Indeed, if $Q_1 \leq Q_2$, then writing

$$G'_{n_0+2}(S_{n_0+2}(Q_1),Q_1) = G'_{n_0+2}(S_{n_0+2}(Q_2),Q_2) = 0,$$

hence

$$G'_{n_0+2}(S_{n_0+2}(Q_1),Q_2) \geq 0,$$

from which it follows that $S_{n_0+2}(Q_1) \geq S_{n_0+2}(Q_2)$.

We now assume that, for $n \geq n_0 + 2$, one has

$$\begin{aligned} &u_n(x,Q) \text{ is convex } C^1, \text{except } x = 0, \\ &u'_n(x,Q) \text{ is increasing in } Q, \forall x \neq 0, u_n(x,Q) \to +\infty, \text{as } |x| \to +\infty \end{aligned}$$

In 0, $u'_n(x,Q)$ has left and right limits, and both limits increase in Q. We also assume

$$u'_n(x,Q) \leq -c - p, \forall x < 0.$$

The function

$$G_n(x,Q) = cx + \alpha E u_n(x-D),$$

is convex, C^1 and

$$G_n(x,Q) \geq cx + \alpha[hE(x-D)^+ + pE(x-D)^- - c(x-\bar{D})],$$

therefore again, from the assumption (11.1.4) we obtain $G_n(x,Q) \to +\infty$ as $|x| \to +\infty$. Moreover $G'_n(x,Q)$ is monotone increasing in Q. It follows that $G_n(x,Q)$ attains its minimum, in a point $S_n(Q)$. Taking the smallest minimum, we can define in a unique way this point. This point is strictly positive, since $G_n'(x,Q) \leq c(1-\alpha) - \alpha p < 0, \forall x \leq 0$. It follows that we can write $u_{n+1}(x,Q)$ as follows

$$u_{n+1}(x,Q) = \begin{cases} hx^+ + px^- - cx + G_n(x+Q,Q), & \text{if } x < S_n(Q) - Q \\ hx^+ + px^- - cx + G_n(S_n(Q),Q), & \text{if } S_n(Q) - Q < x < S_n(Q) \\ hx^+ + px^- - cx + G_n(x,Q) & \text{if } x > S_n(Q) \end{cases}$$

We then check that $u_{n+1}(x,Q)$ has the same properties as $u_n(x,Q)$. Note that $S_n(Q)$ is decreasing in Q.

From the convexity of $u_n(x,Q)$ we deduce that the limit $u(x,Q)$ is convex. Since $u(x,Q) \geq u_n(x,Q)$, we obtain that $u(x,Q) \to +\infty$ as $|x| \to +\infty$. Also $G(x,Q)$ defined by (11.1.5) is convex, continuous and $\to +\infty$ as $|x| \to +\infty$. The function $G(x,Q)$ attains its minimum in a point $S(Q)$ and by taking the smallest minimum, we can define $S(Q)$ in a unique way. The formulas (11.1.6) define an optimal feedback. Let us now prove that $u(x,Q)$ is C^1, except in 0. We first notice the estimate

$$|u_n'(x,Q)| \leq \frac{\max(h,p)+c}{1-\alpha}.$$

Therefore, the limit $u(x,Q)$ is Lipschitz continuous. But then $G(x,Q)$ is C^1, since

$$G'(x,Q) = c + \alpha \int_{-\infty}^{x} u'(z,Q) f(x-z) dz,$$

and $G'(S(Q),Q) = 0$. The function

$$g(x,Q) = \inf_{x \leq \eta \leq x+Q} G(\eta,Q),$$

is also C^1. Therefore

$$u(x,Q) = hx^+ + px^- - cx + g(x,Q),$$

is also C^1, except of 0. Since $G'(0,Q) \leq c(1-\alpha) - \alpha p < 0$, $S(Q) > 0$. We can also check that $S(Q)$ is decreasing and that $u'(x,Q)$ is monotone increasing in Q.

The proof has been completed. □

Remark 11.1. There is no explicit formula for $S(Q)$. It satisfies

$$c + \alpha E u'(S(Q) - D) = 0.$$

However, unlike the case $Q = +\infty$, one cannot express explicitly $Eu'(S(Q) - D)$. Define

$$z(x) = u'(x) - h 1\!\!1_{x>0} + p 1\!\!1_{x<0} + c;$$
$$g(x) = c(1-\alpha) + \alpha h - \alpha(p+h)\bar{F}(x),$$

then the condition becomes

(11.1.10) $$g(S(Q)) + \alpha E z(S(Q) - D) = 0.$$

When $Q = +\infty$,$z(x) = 0, \forall x \leq S$, but it is not true when Q is finite. The function $z(x)$ is the solution of the problem

(11.1.11) $$z(x) = \begin{cases} g(x+Q) + \alpha E z(x+Q-D), & \text{if } x < S(Q) - Q \\ 0 & \text{if } S(Q) - Q < x < S(Q) \end{cases}$$

It suffices to define $z(x)$ for $x \leq S(Q) - Q$. We get the problem

$$z(x) = g(x+Q) + \alpha E z(x+Q-D) 1\!\!1_{D > (x+2Q-S(Q))^+},$$

which is equivalent to the integral equation

$$z(x) = g(x+Q) + \alpha \int_{-\infty}^{\min(x+Q,S(Q)-Q)} z(\eta) f(x+Q-\eta) d\eta. \tag{11.1.12}$$

For each $S > 0$, we solve the integral equation (11.1.12). We then define $S(Q)$ by the algebraic equation (11.1.10).

The case of a set -up cost is open. It is unlikely that the optimal feedback is given by an $s(Q), S(Q)$ policy.

11.2. MULTI SUPPLIER PROBLEM

11.2.1. DESCRIPTION OF THE PROBLEM. We present here a problem considered by E. Porteus, [**33**]. It generalizes the problem of inventory control with set-up cost developed in Chapter 9. For ordering, one supposes now that there are multiple supplier sources. More precisely the ordering cost function is given by

$$C(z) = \begin{cases} 0, & \text{if } z = 0 \\ \min_i (K_i + c_i z) & \text{if } z > 0 \end{cases}, \tag{11.2.1}$$

with

$$c_1 > c_2 > \cdots > c_M > 0, \; 0 < K_1 < K_2 < \cdots < K_M. \tag{11.2.2}$$

The variable costs and the fixed costs vary in the opposite way. So there is not a single supplier which combines the cheapest variable and fixed costs. A trade off is possible. This is, of course, the source of the difficulty.

Exercise 11.1. Show that $C(z)$ is a concave function on $z \geq 0$ and $C(0) = 0$. Show that a function $C(z)$, which is concave on R^+ and vanishes at 0, verifies the sub additivity property

$$C(z_1 + z_2) \leq C(z_1) + C(z_2). \tag{11.2.3}$$

Exercise 11.2. Show the following property

$$C(z) = \min_{\left\{\begin{array}{l} v_1 + \cdots + v_M = z \\ v_i \geq 0 \end{array}\right\}} \sum_{i=1}^{M} (K_i \mathbb{1}_{v_i > 0} + c_i v_i).$$

This is proven by recurrence on M. It can be interpreted as the fact that there is no point splitting the order among the suppliers. One needs to select a single supplier.

We consider the dynamic inventory control problem with backlog. We can easily write the DP equation.

$$u(x) = hx^+ + px^- + \inf_{\begin{array}{c} i = 1, \cdots, M \\ \eta \geq x \end{array}} [K_i \mathbb{1}_{\eta > x} + c_i(\eta - x) + \alpha E u(\eta - D)]. \tag{11.2.4}$$

If we set

$$w(x) = u(x) - hx^+ - px^-, \tag{11.2.5}$$

the D.P. equation is equivalent to

$$w(x) = \inf_{\eta \geq x} [C(\eta - x) + \alpha E(h(\eta - D)^+ + p(\eta - D)^- + w(\eta - D))]. \tag{11.2.6}$$

It is convenient to introduce the following notation. Define

$$g_i(x) = c_i(x + \overline{D}) + \alpha(w(x) + hx^+ + px^-),$$

and

$$G_i(x) = Eg_i(x - D).$$

Let also

$$G_i^*(x) = \min\left[G_i(x), \inf_{y>x}(K_i + G_i(y))\right] = \inf_{y\geq x}(K_i 1\!\!1_{y>x} + G_i(y)),$$

and

$$w_i(x) = -c_i x + G_i^*(x).$$

Finally the DP equation (11.2.6) becomes

$$w(x) = \min_{1\leq i\leq M} w_i(x).$$

The objective is to prove the following result

Theorem 11.3. . *Assume* (11.1.1), (11.1.2) *and*

$$c_1 - \alpha c_M - \alpha p < 0, \tag{11.2.7}$$

and a restriction on the probability density of the demand, namely property $\mathcal{P}$, below (f is Poisson or a convolution of Poisson). Then the feedback $\delta(x)$ in the right hand side of (11.2.4), *is given by a generalized s, S policy. This means that there exists a double sequence*

$$s_m^* < s_{m-1}^* \cdots < s_1^* < S_1 < S_2 \cdots < S_m, \quad 1 \leq m \leq M, \tag{11.2.8}$$

with the feedback

$$\delta(x) = \begin{cases} S_m & \text{if } x \leq s_m^* \\ S_i & \text{if } s_{i+1}^* < x \leq s_i^*, i = 1, \cdots, m-1 \\ x & \text{if } x > s_1^* \end{cases} \tag{11.2.9}$$

11.2.2. AUXILIARY RESULTS. We will need some extension of the K convexity theory. We start with functions $G_i(x)$, satisfying algebraic properties. Later on, we will show that the functions, with the same notation, defined in the previous section satisfy these properties.

Consider functions $G_i(x), i = 1, \cdots, M$, which are continuous, $G_i(x) \to \infty$ as $|x| \to +\infty$. We assume that there exists a_i such that $G_i(x)$ is decreasing on $(-\infty, a_i]$, and verifies

$$G_i(x) < K_i + G_i(y), \forall a_i < x < y.$$

We say that $G_i(x)$ is K_i nondecreasing on $[a_i, +\infty)$. We also assume the relations

$$G_{i+1}(x) - G_i(x) = (c_{i+1} - c_i)x.$$

We set

$$G_i^*(x) = \inf_{y\geq x}(K_i 1\!\!1_{y>x} + G_i(y)),$$

and

$$w_i(x) = -c_i x + G_i^*(x).$$

Finally, we are interested in computing

$$w(x) = \min_{1\leq i\leq M} w_i(x). \tag{11.2.10}$$

The optimization problem is in two steps. For each i find the optimal

$$\widehat{y}_i = \delta_i(x),$$

which achieves the minimum in y in the definition of $G_i^*(x)$. Then find the right index $i(x)$ in the definition of $w(x)$. The feedback function, which expresses the order as a function of the inventory is given by

$$\widehat{y} = \delta(x) = \delta_{i(x)}(x).$$

Beyond the indexing, the framework is reminiscent of K convexity. However, the functions $G_i(x)$ are not K_i convex, but they enjoy a close property, the quasi K_i convexity. Dropping the index, we define the K quasi convexity by

$$G(\theta x + (1-\theta)y) \leq \max[G(x), K + G(y)], \forall x \leq y$$

Exercise 11.3. Show the K quasi convexity property of the functions $G_i(x)$. Show also that a K convex function is K quasi-convex.

Exercise 11.4. Give an example a function which is decreasing in $(-\infty, a]$, K nondecreasing in $[a, \infty)$, and which is not K convex.

The first easy thing is to check that the $\delta_i(x)$ are given by a s_i, S_i policy. Indeed there exist

$$s_i \leq a_i \leq S_i,$$

such that

$$G_i(S_i) = \min_y G_i(y),$$

and

$$G_i(s_i) = K_i + G_i(S_i).$$

Then clearly

$$\delta_i(x) = \begin{cases} S_i, & \text{if } x \leq s_i \\ x, & \text{if } x > s_i \end{cases}$$

Note also the property

$$G_i(x) \leq G_i(s_i), \text{ for } s_i \leq x \leq S_i$$

Remark 11.2. If the function $G_i(x)$ is K_i convex, then we can associate to it, s_i, S_i satisfying the properties above. By taking $a_i = s_i$, we see that the properties of decreasing on $(-\infty, a_i]$ and K_i nondecreasing on $[a_i = \infty)$ are satisfied.

We note that

$$G_i^*(x) = \begin{cases} G_i(s_i), & \text{if } x \leq s_i \\ G_i(x), & \text{if } x \geq s_i \end{cases}$$

So it is a continuous function and $w_i(x)$ is also continuous.

Exercise 11.5. Show that $w(x)$ is also continuous.

One has the property

Lemma 11.1.

$$a_1 \leq S_1 \leq S_2 \leq \cdots \leq S_M.$$

PROOF. Note that, by the definition of the S_i one has

$$G_{i+1}(S_i) - G_i(S_i) \geq G_{i+1}(S_{i+1}) - G_i(S_{i+1}),$$

hence

$$(c_{i+1} - c_i)S_i \geq (c_{i+1} - c_i)S_{i+1},$$

and thus

$$S_i \leq S_{i+1}.$$

□

One has also the property

Lemma 11.2.

If $s_i \leq s_j$, for $i \leq j$ then $w_i(x) \geq w_j(x) \forall x$

PROOF. If $x \leq s_i$ then

$$\begin{aligned} w_i(x) &= -c_i x + G_i(s_i) \\ &= -c_i x + (c_i - c_j)s_i + G_j(s_i) \\ &\geq -c_i x + (c_i - c_j)s_i + G_j(s_j) \\ &= -c_i x + (c_i - c_j)s_i + w_j(x) + c_j x \\ &\geq w_j(x) \end{aligned}$$

If $s_i \leq x \leq s_j$, then

$$\begin{aligned} w_i(x) &= -c_i x + G_i(x) \\ &= -c_j x + G_j(x) \\ &\geq -c_j x + G_j(s_j) \\ &= w_j(x) \end{aligned}$$

Finally

$$\text{If } x \geq s_j, \quad w_i(x) = w_j(x)$$

This conclude the proof. □

It follows that the supplier i can be discarded. So there remains a number $m \leq M$ of suppliers with the following configuration

$$s_m < s_{m-1} < \cdots < s_1 < S_1 \leq \cdots \leq S_m.$$

We now define the concept of generalized s, S policy (see statement of Theorem. It consists in a double sequence

$$s_m^* < s_{m-1}^* \cdots < s_1^* < S_1 < S_2 \cdots < S_m,$$

with the feedback

$$\delta(x) = \begin{cases} S_m, & \text{if } x \leq s_m^* \\ S_i, & \text{if } s_{i+1}^* < x \leq s_i^*, i = 1, \cdots, m-1 \\ x, & \text{if } x > s_1^* \end{cases}$$

Recalling that $\delta(x) = \delta_{i(x)}(x)$, we see that a generalized s, S policy is possible only if

$$s_i^* \leq s_i, s_1^* = s_1.$$

We can formulate the main result of the optimization problem as follows

Theorem 11.4. *The function* $w(x)$ *in* (11.2.10) *is obtained through a generalized* s, S *policy. Namely, there exists* $m \leq M$*, and a subset of* $[1, \cdots, M]$*, which we can renumber in increasing order as* $[1, \cdots, m]$*, and an associated sequence* s_i^**, with* $s_i^* > s_{i+1}^*$*,* $s_i^* \leq s_i$*,* $s_1^* = s_1$*, such that*

$$w(x) = \begin{cases} -c_m x + G_m(s_m), & \text{if } x \leq s_m^*,\ i(x) = m \\ -c_i x + G_i(s_i), & \text{if } s_{i+1}^* < x \leq s_i^*, i = 1, \cdots, m-1,\ i(x) = i \\ -c_1 x + G_1(x), & \text{if } x > s_1^* = s_1, \text{ no order} \end{cases}$$

Proof. Define

$$s_{ij} = \frac{G_i(s_i) - G_j(s_j)}{c_i - c_j};$$

$$s_j^* = \min\left(s_j, \min_{i=1,\cdots,j-1} s_{ij}\right).$$

We shall prove that

$$\text{(11.2.11)} \qquad \begin{array}{l} i(x) \text{ may be chosen } \geq j \text{ if } x \leq s_j^* \\ i(x) \text{ may be chosen } \leq j-1 \text{if } x > s_j^* \end{array}.$$

Indeed, if $x \leq s_j^*$, then

$$w_j(x) = -c_j x + G_j(s_j),$$

and for $i \leq j-1$, $x \leq s_{ij}$ implies

$$x \leq \frac{G_i(s_i) - G_j(s_j)}{c_i - c_j}.$$

Therefore

$$w_j(x) \leq -c_i x + G_i(s_i), \forall i \leq j-1.$$

However

$$w_i(x) = \begin{cases} -c_i x + G_i(s_i), & \text{if } x \leq s_i \\ -c_i x + G_i(x), & \text{if } x > s_i \end{cases}$$

Note also

$$w_i(x) \leq -c_i x + G_i(x), \forall x.$$

Hence if $x \leq s_i$, we can assert that $w_i(x) \geq w_j(x)$. On the other hand, if $x > s_i$, then $w_i(x) = -c_j x + G_j(x) \geq w_j(x)$. So the first part is proven.

Suppose now that $x > s_j^*$, we show that there is some $i \leq j-1$, such that $w_i(x) \leq w_j(x)$. There are several cases. If $s_j^* = s_j$, then $w_j(x) = -c_j x + G_j(x)$, hence

$$w_j(x) = -c_i x + G_i(x) \geq w_i(x) \forall i \neq j.$$

Suppose $s_j^* = \min_{i=1,\cdots,j-1} s_{ij} < s_j$, then if $x \geq s_j$, we have the same situation as before. There remains the case when $s_j^* < x < s_j$. In that case, $w_j(x) = -c_j x + G_j(s_j)$, and there exists $i \leq j-1$, such that $s_{ij} = s_j^* < x$, which implies $w_j(x) > -c_i x + G_i(s_i)$, Moreover $s_i > s_j$, since $i \leq j-1$. Hence $x < s_i$, and $w_i(x) = -c_i x + G_i(s_i)$. We get immediately $w_i(x) < w_j(x)$. So we have proven that there exists $i \leq j-1$, better or equivalent to j.

It follows that

$$\text{if } s_i^* \leq s_j^*, i < j,$$

then the supplier i can never be chosen. Indeed, we know that if $x \leq s_j^*$, we can chose among the suppliers $j, \cdots, M$, hence i may be out. On the other hand if $x > s_j^* \geq s_i^*$, then i as well as j may be excluded. So we eliminate all the indices i, such that there exists $j > i$, with $s_i^* \leq s_j^*$. There remains m indices, with decreasing

s_i^*. We may renumber them and define in this way the sequence $s_1^* > s_2^* \cdots > s_m^*$. It follows that if $s_{i+1}^* < x \leq s_i^*$, then $i(x)$ can be chosen among the indices larger or equal to i, but we may exclude $i+1, \cdots, m$, hence i is optimal. If $x > s_1^* = s_1$, we may eliminate all indices, hence it is optimal not to order.

The proof has been completed. □

We now prove some properties of the function w

Corollary 11.1. *The function $w(x)$ is decreasing on $(-\infty, a_1]$, and $w(x) + c_1 x$ is K_1 non decreasing on $[a_1, +\infty)$.*

Proof. Recall that $w(x)$ is continuous. From the formula proved in Lemma 11.4 it follows that $w(x)$ decreases on $(-\infty, s_1)$ and from the properties of G_1 also decreasing on (s_1, a_1). The second part is obvious. □

Let us check additional properties of w.

Corollary 11.2. *$w(x) + c_i x$ is K_i nondecreasing on the whole line R. Assume $a_m \geq 0$, then $w(x) + c_m x$ is decreasing on $(-\infty, 0]$.*

Proof. Noting that

$$G_i(x) = G_1(x) + (c_i - c_1)x.$$

We can write

$$G_i^*(x) = \inf_{y \geq x} (K_i \mathbb{1}_{y>x} + G_1(y) - c_1 y + c_i y).$$

Since

$$w_i(x) = -c_i x + G_i^*(x),$$

it follows that

$$w_i(x) = inf_{y \geq x}[K_i \mathbb{1}_{y>x} + c_i(y-x) + G_1(y) - c_1 y)].$$

It follows that

$$w(x) = inf_{z \geq x}(C(z-x) + G_1(z) - c_1 z),$$

where

$$C(z) = \min_{i=1}^{m} [K_i \mathbb{1}_{z>0} + c_i z].$$

Take $y \geq x$, then

$$w(x) \leq \inf_{z \geq y} [C(z-x) + G_1(z) - c_1 z)],$$

and from the subadditivity of $C(z)$, and $z \geq y \geq x$, it follows

$$w(x) \leq w(y) + C(y-x),$$

from which we deduce

$$w(x) \leq w(y) + K_i + c_i(y_i - x_i), \forall i.$$

Hence $w(x) + c_i x$ is K_i nondecreasing on the whole line R. Next

$$w(x) + c_m x = \begin{cases} G_m(s_m), & \text{if } x \leq s_m^* \\ (c_m - c_i)x + G_i(s_i), & \text{if } s_{i+1}^* < x \leq s_i^* \\ (c_m - c_1)x + G_1(x), & \text{if } x > s_1^* \end{cases}$$

In fact, one observes that

$$w(x) + c_m x = G_m(x), \text{ if } x > s_1^*.$$

From the above expression, using $c_m \leq c_i, \forall i$, and the fact that $a_m \geq 0$, hence $G_m(x)$ is decreasing on $(-\infty, 0]$, the second part of Corollary 11.2 follows. □

11.2.3. APPLICATION TO THE MULTI_SUPPLIER PROBLEM. To apply the preceding results to the multi-supplier problem, we need an additional intermediary result. Consider a function $\psi(x)$, piece wise continuous, which tends to $+\infty$ as $|x| \to +\infty$, hence attains its minimum. Let D be a positive random variable, with density f. The quantity

$$\chi(x) = E\psi(x - D),$$

is a continuous function, which also tends to $+\infty$ as $x \to +\infty$.

Exercise 11.6. Prove the preceding property. Use the fact that $\psi(x)$ reaches its minimum.

However, if ψ decreases on $(-\infty, a]$, and is non K decreasing, on $[a, +\infty)$, then this property does not carry over to χ. The same bad situation occurs whenever ψ is quasi K convex. In the case of K convexity, the property carries over. Note however, that $\chi(x)$ decreases on $(-\infty, a]$, and since it goes to $+\infty$, as $x \to +\infty$, there is necessarily some $b \geq a$, such that $\chi(x)$ decreases on $(-\infty, b]$, and may increase after. So b is a local minimum. We want to restrict ourselves to distributions f such that the following property $\mathcal{P}$ holds:

The function χ is non K decreasing on $[b, +\infty)$.

Let us show that the property $\mathcal{P}$ holds, whenever the distribution f is Poisson, or a convolution of Poisson distributions. Suppose f is a Poisson distribution, with parameter λ, then

$$f(x) = \lambda \exp -\lambda x.$$

Exercise 11.7. Check that

$$\chi'(x) = \lambda(\psi(x) - \chi(x)).$$

Let S be the minimum of χ. Since $\chi'(b) = \chi'(S) = 0$, we have $\chi(S) = \psi(S)$ and $\chi(b) = \psi(b)$. Let $h > 0$. Consider the function

$$z(x) = \chi(x) - \chi(x + h) - K,$$

then we have

$$z(b) \leq \chi(b) - \chi(S) - K = \psi(b) - \psi(S) - K \leq 0,$$

since $a \leq b \leq S$.

Exercise 11.8. Check that

$$z'(x) \leq -\lambda z(x).$$

Therefore

$$\frac{dz \exp \lambda x}{dx} \leq 0,$$

and it follows that

$$z(x) \exp \lambda x \leq z(b) \exp \lambda b, \forall x \geq b,$$

hence $z(x) \leq 0, \forall x \geq b$. This implies that χ is K nondecreasing for $x \geq b$. The property $\mathcal{P}$ carries over to convolutions of Poisson distributions. Indeed, if

$$D = \sum_{j=1}^{J} D_j,$$

where the random variables D^j are independent and have Poisson distributions (with parameters λ^j), then define

$$\gamma^j(x) = E\psi\left(x - \sum_{i=1}^{j} D_i\right),\ \gamma^0(x) = \psi(x).$$

Clearly $\chi(x) = \gamma^J(x)$.

Exercise 11.9. Prove that

$$\gamma^j(x) = E\gamma^{j-1}(x - D_j).$$

Since each of the demands has a Poisson distribution, one can checks recursively, that there exists a b such that $\chi(x)$ is decreasing on $(-\infty, b]$, and K nondecreasing on $[b, +\infty)$.

We are in a position to solve the multi supplier problem. It is convenient to introduce the operator, for which w in equation (11.2.6) is a fixed point. More precisely, introduce functions $v(x)$ such that

$$\begin{array}{l} v \text{ is continuous, } (c_M + \alpha h)x + \alpha v(x) \to +\infty, \text{ as } x \to +\infty \\ (c_1 - \alpha p)x + \alpha v(x) \to +\infty, \text{ as } x \to -\infty \\ v(x) + c_M x \text{ is decreasing on } (-\infty, 0] \\ v(x) + c_i x \text{ is } K_i \text{ nondecreasing on } R \end{array} \tag{11.2.12}$$

Note that, from the third condition, we have also

$$v(x) \to +\infty, \text{ as } x \to -\infty.$$

We introduce the operator which associates to the function $v(x)$, successively the functions

$$g_i(v)(x) = c_i(x + \overline{D}) + \alpha(hx^+ + px^-) + \alpha v(x),$$

and

$$G_i(v)(x) = Eg_i(v)(x - D).$$

Let also

$$G_i^*(v)(x) = \inf_{y \geq x}(K_i 1\!\!\mathrm{I}_{y>x} + G_i(v)(y)),$$

and

$$w_i(v)(x) = -c_i x + G_i^*(v)(x),$$

$$w(v)(x) = \min_{1 \leq i \leq M} w_i(v)(x).$$

Then, we can assert that

$$w(x) = w(w)(x).$$

We begin with the

Lemma 11.3. *Assume* (11.2.7) *and that the demand D satisfies the property $\mathcal{P}$. Skipping v in the notation of $G_i(v)$, we can state that*

$$\begin{array}{l} G_i \textit{ is continuous, } G_i \to +\infty, \textit{ as } |x| \to +\infty \\ \exists a_i \geq 0 \textit{ such that } G_i \textit{ is decreasing on } (-\infty, a_i] \\ G_i \textit{ is non } K_i \textit{ decreasing on } [a_i, +\infty) \end{array} . \tag{11.2.13}$$

PROOF. Writing

$$g_i(x) = c_i\overline{D} + (c_i - c_1)x + (c_1 - \alpha c_M)x + \alpha(hx^+ + px^-) + \alpha(c_M x + v(x)),$$

we see immediately from the assumption (11.2.7) and the properties of v that

$$g_i(x) \text{ is decreasing on } (-\infty, 0].$$

Writing also

$$g_i(x) = c_i\bar{D} + (1-\alpha)c_i x + \alpha(hx^+ + px^-) + \alpha(c_i x + v(x)),$$

we get from the properties of v that

$$g_i(x) \text{ is non } K_i \text{ decreasing on } [0, +\infty).$$

Clearly $g_i(x) \to +\infty$, as $|x| \to +\infty$. It follows that $G_i(x)$ is continuous, by the smoothing effect of the mathematical expectation, $G_i(x) \to +\infty$, as $|x| \to +\infty$, and since the demand satisfies property $\mathcal{P}$, there exist $a_i \geq 0$, such that (11.2.13) is verified.

From the general analysis of the intermediary optimization problem, considered earlier, and in particular Corollary 11.2, it follows that $w(v)$ verifies the same property as v, namely (11.2.12). Moreover the optimal feedback denoted $\delta(v)(x)$ is given by a generalized s, S policy.

We apply the developments above to the multi supplier problem. We start with $w_0(x) = -c_M x$. If we assume (11.2.7), then this function satisfies the properties of the generic function v in (11.2.12). So the recursion can start. This completes the proof of Theorem 11.3. □

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CHAPTER 12

INVENTORY CONTROL WITH MARKOV DEMAND

12.1. INTRODUCTION

In all models considered so far, the demand has been treated as a sequence of independent identically distributed random variables. In reality, the successive demands are linked for a lot of reasons. The simplest way to model this linkage is to assume that the demands form a Markov process as such or are derived from a Markov process. Our objective here is to show how the methods described in the preceding chapters can be adapted to this situation. In a recent book by D. Beyer et al. **[12]** a comprehensive presentation of these problems is given. We refer also to the references of this book for the related literature. In this work, the authors consider that the demand comes from an underlying state of demand, which is modeled as a Markov chain with a finite number of states. The fact that the number of states is finite simplifies mathematical arguments. We will here discuss the situation in which the demand itself is a Markov process.

12.2. NO BACKLOG AND NO SET-UP COST

12.2.1. THE MODEL. Let $\Omega, \mathcal{A}, P$ be a probability space, on which is defined a Markov chain z_n. This Markov chain represents the demand. Its state space is R^+ and its transition probability is $f(\zeta|z)$. We shall assume that

(12.2.1) $\quad f(\zeta|z)$ is uniformly continuous in both variables and bounded

$$\int_0^{+\infty} \zeta f(\zeta|z) d\zeta \leq c_0 z + c_1. \tag{12.2.2}$$

We can assume that $z_1 = z$, a fixed constant or more generally a random variable with given probability distribution. We define the filtration

$$\mathcal{F}^n = \sigma(z_1, \cdots, z_n).$$

A control policy, denoted by V, is a sequence of random variables v_n such that v_n is $\mathcal{F}^n$ measurable. When $z_1 = z$, then v_1 is deterministic. Also, we assume as usual that $v_n \geq 0$. We next define the inventory as the sequence, with

$$y_{n+1} = (y_n + v_n - z_{n+1})^+, \quad y_1 = x. \tag{12.2.3}$$

The process y_n is adapted to the filtration $\mathcal{F}^n$. The joint process y_n, z_n is also a Markov chain.

We can write, for a test function $\varphi(x, z)$ (bounded continuous on $R^+ \times R^+$)

$$\begin{aligned} &E[\varphi(y_2, z_2)|y_1 = x, z_1 = z, v_1 = v] \\ &\quad = E[\varphi((x + v - z_2)^+, z_2)|z_1 = z] \\ &\quad = \int_{x+v}^{+\infty} \varphi(0, \varsigma) f(\zeta|z) d\zeta + \int_0^{x+v} \varphi(x + v - \zeta, \zeta) f(\zeta|z) d\varsigma \end{aligned}$$

This defines a Markov chain to which is associated the operator

$$\Phi^v \varphi(x, z) = \int_{x+v}^{+\infty} \varphi(0, \zeta) f(\zeta|z) d\zeta + \int_0^{x+v} \varphi(x + v - \zeta, \zeta) f(\zeta|z) d\varsigma, \tag{12.2.4}$$

hence the transition probability is given by

$$\begin{aligned} \pi(x, z, v; d\xi, d\zeta) = &\ \delta(\xi) \otimes f(\varsigma|z) \mathbb{1}_{\varsigma > x+v} d\zeta \\ &+ \delta(\xi - (x + v - \zeta)) \otimes f(\zeta|z) \mathbb{1}_{\varsigma < x+v} d\zeta. \end{aligned} \tag{12.2.5}$$

We next define the function

$$l(x, z, v) = cv + hx + p \int_{x+v}^{\infty} (\zeta - x - v) f(\zeta|z) d\zeta, \tag{12.2.6}$$

which will model the one period cost. The cost function is then

$$J_{x,z}(V) = E \sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, z_n, v_n). \tag{12.2.7}$$

We are interested in the value function

$$u(x, z) = \inf_V J_{x,z}(V). \tag{12.2.8}$$

We first notice that the properties (4.3.1), (4.3.2) are satisfied. To proceed, we need to specify a ceiling function. It corresponds to a control identically 0. Set $l_v(x, z) = l(x, z, v)$. Note the inequalities

$$cv + hx \leq l_v(x, z) \leq cv + hx + p(c_0 z + c_1), \tag{12.2.9}$$

and

$$cv \leq \Phi^v l_v(x, z) \leq cv + h(x + v) + p(c_0^2 z + c_0 c_1 + c_1). \tag{12.2.10}$$

Consider then the function

$$w_0(x, z) = \sum_{n=1}^{\infty} \alpha^{n-1} (\Phi^0)^{n-1} l_0(x, z) \tag{12.2.11}$$

Lemma 12.1. *We assume that*

$$\alpha c_0 < 1, \tag{12.2.12}$$

then the series $w_0(x, z) < \infty$, *and more precisely*

$$w_0(x, z) \leq \frac{hx}{1 - \alpha} + \frac{pc_0 z}{1 - c_0 \alpha} + \frac{pc_1}{(1 - \alpha)(1 - c_0 \alpha)}. \tag{12.2.13}$$

PROOF. It is an immediate consequence of formulas (12.2.11) and (12.2.10). □

We can now write the Bellman equation

$$u(x, z) = \inf_{v \geq 0} [l(x, z, v) + \alpha \Phi^v u(x, z)]. \tag{12.2.14}$$

We state the following

Theorem 12.1. *We assume* (12.2.1), (12.2.2), (12.2.6), (12.2.12). *Then there exists one and only one solution of equation* (12.2.14), *such that* $0 \le u \le w_0$. *It is continuous and coincides with the value function* (12.2.8). *There exists an optimal feedback* $\hat{v}(x,z)$ *and an optimal control policy* $\hat{V}$.

PROOF. We first notice that conditions of Theorem 4.4 and 4.3 are satisfied. Hence the set of solutions of equation (12.2.14) is not empty. It has a minimum and a maximum solution. The minimum solution is l.s.c. and the maximum solution is u.s.c. The minimum solution coincides with the value function. There exists an optimal feedback $\hat{v}(x,z)$ and an optimal control policy $\hat{V}$. To prove uniqueness, we have to prove that the minimum and maximum solutions coincide. We begin by proving a bound on $\hat{v}(x,z)$. It will be first convenient to mention a slightly better estimate for w_0. Indeed, we can write

$$w_0(x,z) \le h\sum_{n=1}^{\infty}(\alpha\Phi^0)^{n-1}x(x,z) + \frac{pc_0 z}{1-c_0\alpha} + \frac{pc_1}{(1-\alpha)(1-c_0\alpha)}, \tag{12.2.15}$$

and (12.2.13) was simply derived from (12.2.15) by using $\Phi^0 x(x,z) \le x$. Next, from (12.2.14), considering $\underline{u}$, the minimum solution, we can state that

$$\underline{u}(x,z) \ge h\sum_{n=1}^{\infty}(\alpha\Phi^0)^{n-1}x(x.z),$$

which follows from $l(x,z,v) \ge hx$, and

$$\Phi^v\varphi(x,z) \ge \Phi^0\varphi(x,z), \forall v \ge 0, \forall\varphi \text{ increasing in } x. \tag{12.2.16}$$

Therefore, we can write

$$l(x,z,v) + \alpha\Phi^v\underline{u}(x,z) \ge cv + h\sum_{n=1}^{\infty}(\alpha\Phi^0)^{n-1}x(x,z).$$

Therefore, in minimizing in v, we can bound from above the range of v. More precisely, we get

$$\hat{v}(x,z) \le \frac{pc_0 z}{c(1-c_0\alpha)} + \frac{pc_1}{c(1-\alpha)(1-c_0\alpha)}. \tag{12.2.17}$$

We consider the optimal trajectory $\hat{y}_n, z_n$ obtained from the optimal feedback, namely

$$\hat{y}_{n+1} = (\hat{y}_n + \hat{v}_n - z_{n+1})^+,\ \hat{v}_n = \hat{v}(\hat{y}_n, z_n)$$

with $\hat{y}_1 = x, z_1 = z$.

Following section 4.5 of Chapter 4, the maximum solution will coincide with the minimum solution, if we can check that $\hat{V} \in \mathcal{V}$, i.e.

$$\alpha^n E\bar{u}(\hat{y}_{n+1}, z_{n+1}) \to 0, \text{ as } n \to \infty$$

It is sufficient to replace $\bar{u}$ by w_0 and by the estimate (12.2.13), it is sufficient to show that

$$\alpha^n E\hat{y}_{n+1}, \quad \alpha^n E z_{n+1} \to 0, \text{ as } n \to \infty. \tag{12.2.18}$$

However, by standard Markov arguments, $\alpha^n E z_{n+1} \le (\alpha c_0)^n z$. From the assumption (12.2.12), the second part of (12.2.18) follows immediately. We next use

$$\begin{aligned} \hat{y}_{n+1} &\le \hat{y}_n + \hat{v_n} \\ &\le \hat{y}_n + \frac{pc_0 z_n}{c(1-c_0\alpha)} + \frac{pc_1}{c(1-\alpha)(1-c_0\alpha)}. \end{aligned}$$

Therefore

$$E\hat{y}_{n+1} \leq E\hat{y}_n + \frac{pc_0^n z}{c(1-c_0\alpha)} + \frac{pc_1}{c(1-\alpha)(1-c_0\alpha)},$$

and we deduce the estimate

$$E\hat{y}_{n+1} \leq x + \frac{pzc_0}{c(1-\alpha c_0)}\frac{1-c_0^n}{1-c_0} + \frac{npc_1}{c(1-\alpha)(1-c_0\alpha)}.$$

Using again the assumption (12.2.12), we deduce the first part of (12.2.18). This completes the proof. □

12.2.2. BASE-STOCK POLICY. We want now to check that the optimal feedback $\hat{v}(x,z)$ can be obtained by a base stock policy, with a base stock depending on the value of z. We have the

Theorem 12.2. *We make the assumptions of Theorem 12.1 and $p > c$. We assume also*

(12.2.19) $$f(x|z) \geq a_0(M) > 0, \; \forall x, z \leq M.$$

Then the function $u(x,z)$ is convex and C^1 in the argument x. Moreover the optimal feedback is given by

(12.2.20) $$\hat{v}(x,z) = \begin{cases} S(z) - x, & \text{if } x \leq S(z) \\ 0, & \text{if } x \geq S(z) \end{cases}$$

The function $S(z)$ is uniformly continuous and the derivative in x, $u'(x,z)$ is uniformly continuous.

Proof. We consider the increasing process

$$u_{n+1}(x,z) = \inf_{v \geq 0}[l(x,z,v) + \alpha\Phi^v u_n(x,z)], \qquad u_0(x) = 0$$

which we write also as

$$u_{n+1}(x,z) = (h-c)x + \inf_{\eta \geq x}\{c\eta + pE[(\eta - z_2)^-|z_1 = z] + \alpha E[u_n((\eta - z_2)^+, z_2)|z_1 = z]\},$$

and we are going to show recursively that the sequence $u_n(x,z)$ is convex and C^1 in x. Define

$$G_n(x,z) = cx + pE[(x - z_2)^-|z_1 = z] + \alpha E[u_n((x - z_2)^+, z_2)|z_1 = z],$$

then the function $G_n(x,z)$ attains its minimum in $S_n(z)$, which can be uniquely defined by taking the smallest minimum. We first consider

$$G_0(x,z) = cx + pE[(x - z_2)^-|z_1 = z].$$

It is convex and C^1 in x. We have

$$G_0'(x,z) = c - p\bar{F}(x|z),$$

with

$$\bar{F}(x|z) = \int_x^\infty f(\xi|z)d\xi.$$

The function $G_0'(x,z)$ increases in x, from $c-p<0$ to c. Thanks to the assumption, we define in a unique way $S_0(z)$ such that $G_0'(S_0,z)=0$. From convexity, we can assert that

$$u_1(x,z)=\begin{cases}(h-c)x+G_0(S_0(z),z), & \text{if } x\le S_0(z)\\(h-c)x+G_0(x,z), & \text{if } x\ge S_0(z)\end{cases}$$

We see that $u_1(x,z)$ is convex and C^1 in x. Then

$$\begin{aligned}G_1(x,z)&=cx+pE[(x-z_2)^-|z_1=z]+\alpha(h-c)E[(x-z_2)^+|z_1=z]\\&\quad+\alpha E[G_0(\max(x-z_2,S_0(z_2)),z_2)|z_1=z]\end{aligned}$$

and we see that $G_1(x,z)$ is convex and C^1 in x. By induction we check that $u_n(x,z)$, $G_n(x,z)$ are convex and C^1 in x.

Moreover

$$G_n'(x,z)=c-p\bar F(x|z)+\alpha E[u_n'(x-z_2,z_2)1\!\!1_{x\ge z_2}|z_1=z]$$

We see that $G_n'(0,z)=c-p$ and we can check that $G_n'(+\infty,z)=c+h\alpha\dfrac{1-\alpha^n}{1-\alpha}$. Therefore there exists a point $S_n(z)$ such that $G_n'(S_n(z),z)=0$. Note that $S_n(z)>0$, since $G_n'(0,z)=c-p<0$. We have

$$u_{n+1}(x,z)=\begin{cases}(h-c)x+G_n(S_n(z),z), & \text{if } x\le S_n(z)\\(h-c)x+G_n(x,z), & \text{if } x\ge S_n(z)\end{cases}$$

It follows that the limit $u(x,z)$ is convex in x. Clearly $u(x,z)\to+\infty$, as $x\to+\infty$. Also

$$G(x,z)=cx+pE[(x-z_2)^-|z_1=z]+\alpha E[u((x-z_2)^+,z_2)|z_1=z]$$

is convex and $\to+\infty$ as $x\to+\infty$. So The minimum is attained in $S(z)$, which can be defined in a unique way, by taking the smallest minimum.

Since $h-c\le u_n'(x,z)\le\dfrac{h}{1-\alpha}$, we can assert that $u(x,z)$ is Lipschitz continuous in x. The same is true for $G(x,z)$. But

$$G'(x,z)=c-p\bar F(x|z)+\alpha E[u'(x-z_2,z_2)1\!\!1_{x>z_2}|z_1=z]\to c-p \text{ as } x\to 0$$

therefore also $S(z)>0$. Then, from convexity

$$u(x,z)=\begin{cases}(h-c)x+G(S(z),z), & \text{if } x\le S(z)\\(h-c)x+G(x,z), & \text{if } x\ge S(z)\end{cases}$$

Next, from

$$c-p\bar F(S_n(z)|z)+\alpha E[u_n'(S_n(z)-z_2,z_2)1\!\!1_{S_n(z)>z_2}|z_1=z]=0$$

we deduce, using the estimate on u_n' and the property

$$\bar F(S_n(z)|z)\le\frac{c_0z+c_1}{S_n(z)}$$

that

$$S_n(z)\le\frac{(c_0z+c_1)(p+\alpha(h-c))}{c+\alpha(h-c)}. \tag{12.2.21}$$

The same estimate holds for $S(z)$. It is easy to check that $S_n(z)\to S(z)$, and $S(z)$ is the smallest minimum of $G(x,z)$ in x. Furthermore, from the continuity properties of $G(x,z)$ in both arguments, we can check that $S(z)$ is a continuous function. The feedback $\hat{v}(x,z)$ defined by (12.5.12) is also continuous in both arguments. Define

$$\chi(x,z) = u'(x,z) - h + c$$

as an element of B (space of measurable bounded functions on $R^+ \times R^+$), then χ is the unique solution in B of the equation

$$\begin{aligned}\chi(x,z) &= g(x+\hat{v}(x,z),z)\\ &+\alpha E[\chi(x+\hat{v}(x,z)-z_2,z_2)\mathbb{1}_{x+\hat{v}(x,z)>z_2}|z_1=z],\ x,z\in R^+,\end{aligned} \tag{12.2.22}$$

where

$$g(x,z) = c + \alpha(h-c) - (p+\alpha(h-c))\bar{F}(x|z).$$

Since the function $g(x+\hat{v}(x,z),z)$ is continuous in the pair x,z, the solution $\chi(x,z)$ is also continuous.

Let us check that $S(z)$ is uniformly continuous. Let us first check that

$$(G'(x_1,z)-G'(x_2,z))(x_1-x_2) \geq (p-\alpha(c-h))(F(x_1|z)-F(x_2|z))(x_1-x_2). \tag{12.2.23}$$

Assume, to fix the ideas that $x_1 > x_2$. We have

$$\begin{aligned}(G'(x_1,z)-G'(x_2,z))(x_1-x_2) &= p(F(x_1|z)-F(x_2|z))(x_1-x_2)\\ &+\alpha E[(u'(x_1-z_2,z_2)-u'(x_2-z_2,z_2))(x_1-x_2)\mathbb{1}_{z_2<x_2}|z_1=z]\\ &+\alpha E[u'(x_1-z_2,z_2)\mathbb{1}_{x_2<z_2<x_1}(x_1-x_2)|z_1=z]\end{aligned}$$

The second term is positive, from the convexity of u. Using the left estimate on u' for the last term, we deduce (12.2.23). We then obtain

$$\begin{aligned}&(S(z)-S(z'))(G'(S(z),z')-G'(S(z'),z))\\ &\geq (p-\alpha(c-h))(S(z)-S(z'))\left[\int_{S(z')}^{S(z)}(f(\xi|z)+f(\xi|z'))d\xi\right]\end{aligned}$$

If $z,z' < m$, then from (12.2.21) we can find $M_m > m$ such that $S(z), S(z') < M_m$. From the assumption (12.2.19), we deduce

$$\begin{aligned}&(S(z)-S(z'))(G'(S(z),z')-G'(S(z'),z))\\ &\geq 2a_0(M_m)(p-\alpha(c-h))(S(z)-S(z'))^2\end{aligned}$$

Next we have

$$G'(x,z')-G'(x,z) = \int_0^x (p+\alpha u'(x-\zeta,\zeta))(f(\zeta|z')-f(\zeta|z))d\zeta,$$

hence

$$|G'(x,z')-G(x,z)| \leq xC \sup_{0<\zeta<x}|f(\zeta|z')-f(\zeta|z)|.$$

Applying this estimate with $x = S(z)$ and $x = S(z')$ and combining estimates, we obtain easily that $S(z)$ is uniformly continuous. It follows that the feedback $\hat{v}(x,z)$ is uniformly continuous. Then from (12.2.22), we obtain that $\chi(x,z)$ is uniformly continuous. The proof has been completed. □

Remark 12.1. We have $\chi(x,z) = 0, \forall x \leq S(z)$, and

$$\chi(x,z) = g(x,z) + \alpha E[\chi(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1=z],\ \forall x \geq S(z) \tag{12.2.24}$$

Also

$$0 = g(S(z),z) + \alpha E[\chi(S(z)-z_2,z_2)\mathbb{1}_{S(z)>z_2}|z_1=z]. \tag{12.2.25}$$

So $S(z)$ is not the solution of $g(S(z), z) = 0$. If we consider the function

$$G^*(x,z) = (p + \alpha(h-c))E[(x-z_2)^+|z_1 = z] - (p-c)x + pE[z_2|z_1 = z],$$

then the solution of $g(S, z) = 0$ is denoted by $S^*(z)$. We have $G(x,z) \geq G^*(x,z)$.

12.2.3. ERGODIC THEORY. We turn now to the case when $\alpha \to 1$. We write $u_\alpha(x,z)$ to satisfy
(12.2.26)

$$u_\alpha(x,z) = \begin{cases} (h-c)x + cS_\alpha(z) + pE[(S_\alpha(z)-z_2)^-|z_1 = z]+ \\ \quad +\alpha E[u_\alpha((S_\alpha(z)-z_2)^+, z_2)|z_1 = z], & \text{if } x \leq S_\alpha(z) \\ hx + pE[(x-z_2)^-|z_1 = z] \\ \quad +\alpha E[u_\alpha((x-z_2)^+, z_2)|z_1 = z], & \text{if } x \geq S_\alpha(z) \end{cases}$$

We shall make the assumptions

$$c_0 = 0, \text{ and} \qquad (12.2.27)$$

$$\inf_{\substack{0 \leq \zeta \leq a \\ z}} f(\zeta|z) \geq \gamma(a) > 0, \forall a$$

$$f(\zeta|z) \text{ is ergodic} \qquad (12.2.28)$$

$$\int |f(\zeta|z) - f(\zeta|z')|d\zeta \leq \delta|z-z'| \qquad (12.2.29)$$

$$\sup_z F(x|z) = \delta_0(x) < 1, \forall x$$

We denote by $\varpi(z)$ the invariant probability density corresponding to the Markov chain $f(\zeta|z)$.

We state the

Theorem 12.3. *We assume* (12.2.1), (12.2.2) *with* $c_0 = 0$*,* (12.2.6) *with* $p > c$ *and* (12.2.27), (12.2.28), (12.2.29). *Then, for a subsequence (still denoted* α*) converging to* 1*, we have, for any compact* K *of* R^+

$$\sup_{z \in K} |S_\alpha(z) - S(z)| \leq \epsilon(\alpha, K), \quad \epsilon(\alpha, K) \to 0, \text{ as } \alpha \uparrow 1 \qquad (12.2.30)$$

$$\sup_{\substack{x \leq M \\ z \leq N}} \left| u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z) \right| \to 0, \forall M, N, \qquad (12.2.31)$$

with $\rho_\alpha \to \rho$ *and*
(12.2.32)

$$u(x,z) + \rho = \begin{cases} (h-c)x + cS(z) + pE[(S(z)-z_2)^-|z_1 = z] \\ \quad +E[u((S(z)-z_2)^+, z_2)|z_1 = z] & \text{if } x \leq S(z) \\ hx + pE[(x-z_2)^-|z_1 = z] \\ \quad +E[u((x-z_2)^+, z_2)|z_1 = z] & \text{if } x \geq S(z) \end{cases}$$

The function $u(x,z)$ *satisfies the growth condition*

$$\sup_{\substack{\xi \leq x \\ z}} |u(\xi,z)| \leq C_0 + \frac{C_1 x}{1-\delta_0(x)}. \qquad (12.2.33)$$

It is C^1 in x and Lipschitz continuous in z. The following estimates hold

$$\sup_{\substack{\xi \leq x \\ z}} |u_x(\xi, z)| \leq \frac{C}{1-\delta_0(x)}, \tag{12.2.34}$$

$$|u(x,z) - u(x,z')| \leq \left[C_0(m_0) + \frac{C_1(m_0)x}{1-\delta_0(x)}\right] \delta|z-z'|$$

$$(1-\delta_0(x)) \sup_{\substack{\xi \leq x \\ z}} |u_{xx}(\xi, z)| \leq C, \; (1-\delta_0(x)) \sup_{\substack{\xi \leq x \\ z}} |u_{xz}(\xi, z)| \leq C, \; a.e. \tag{12.2.35}$$

Given $S(z)$, the pair $u(x,z), \rho$ satisfying the above conditions and $\int u(0,z)\varpi(z)dz = 0$ is uniquely defined.

Also

$$u(x,z) + \rho = \inf_{v \geq 0}[l(x,z,v) + \Phi^v u(x,z)]. \tag{12.2.36}$$

PROOF. We begin with (12.2.30). We first note that, thanks to $c_0 = 0$, we have

$$S_\alpha(z) \leq \frac{c_1(p+h)}{\min(h,c)} = m_0. \tag{12.2.37}$$

We pursue the estimates obtained in Theorem 12.2. We have, first

$$\begin{aligned}&(S_\alpha(z) - S_\alpha(z'))(G'_\alpha(S_\alpha(z), z') - G'_\alpha(S_\alpha(z'), z)) \\ &\geq 2\min(h,c)\gamma(m_0)(S_\alpha(z) - S_\alpha(z'))^2\end{aligned}$$

We know that $u_\alpha(x,z)$ is C^1 in x. Then, from (12.2.26), we have (denoting $u'_\alpha(x,z) = u'_{\alpha x}(x,z)$

$$\begin{aligned} u'_\alpha(x,z) &= h - c \quad \text{if} \quad x < S_\alpha(z) \\ &= h - p\bar{F}(x|z) + \alpha E[u'_\alpha(x-z_2, z_2)1\!\!1_{x>z_2}|z_1 = z], \text{ if } x > S_\alpha(z)\end{aligned}$$

from which we can assert that

$$\sup_{\substack{\xi \leq x \\ z}} |u'_\alpha(\xi, z)| \leq \frac{\max(h,p)}{1-\delta_0(x)}. \tag{12.2.38}$$

Therefore

$$|G'_\alpha(S_\alpha(z), z') - G'_\alpha(S_\alpha(z), z)| = \left(p + \frac{\max(h,p)}{1-\sup_z F(m_0|z)}\right)\int |f(\zeta|z') - f(\zeta|z)|d\zeta,$$

$$|G'_\alpha(S_\alpha(z'), z') - G'_\alpha(S_\alpha(z'), z)| = \left(p + \frac{\max(h,p)}{1-\sup_z F(m_0|z)}\right)\int |f(\zeta|z') - f(\zeta|z)|d\zeta.$$

Collecting results we obtain the estimate

$$|S_\alpha(z) - S_\alpha(z')| \leq \frac{p + \dfrac{\max(h,p)}{1-\delta_0(m_0)}}{\min(h,c)\gamma(m_0)} \int |f(\zeta|z') - f(\zeta|z)|d\zeta, \tag{12.2.39}$$

and from the first assumption (12.2.29), we finally obtain

$$|S_\alpha(z) - S_\alpha(z')| \leq \frac{p + \dfrac{\max(h,p)}{1-\delta_0(m_0)}}{\min(h,c)\gamma(m_0)}\delta|z-z'|. \tag{12.2.40}$$

Therefore, the sequence $S_\alpha(z)$ is uniformly Lipschitz continuous. It is standard (see Appendix A.4) that one can extract a subsequence, which converges in the space of continuous functions on compact sets, for any compact set K, towards a function $S(z)$. Therefore (12.2.30) is satisfied.

Define $\chi_\alpha(x,z) = u'_\alpha(x,z) - h + c$. We have

$$\begin{aligned} \chi_\alpha(x,z) &= g_\alpha(x,z) \\ &\quad + \alpha E[\chi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1 = z],\ x > S_\alpha(z) \\ &= 0, \qquad x \leq S_\alpha(z) \end{aligned} \tag{12.2.41}$$

with

$$g_\alpha(x,z) = (c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x|z). \tag{12.2.42}$$

As it has been done for $u'_\alpha(x,z)$, we can state

$$\sup_{\substack{0\leq\xi\leq x \\ z}} |\chi_\alpha(\xi,z)| \leq \frac{\max(h,p)}{1-\delta_0(x)}. \tag{12.2.43}$$

If we consider $\psi_\alpha(x,z) = \chi'_\alpha(x,z) = u"_\alpha(x,z)$, then using the fact that $\chi_\alpha(0,z) = 0$, we see that

$$\begin{aligned} \psi_\alpha(x,z) &= (p + \alpha(h-c))f(x|z) \\ &\quad + \alpha E[\psi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1 = z], \quad x > S_\alpha(z) \\ &= 0, \quad x < S_\alpha(z) \end{aligned} \tag{12.2.44}$$

The function $\psi_\alpha(x,z)$ is not continuous, however it is measurable and bounded. We have

$$\sup_{\substack{0\leq\xi\leq x \\ z}} |\psi_\alpha(\xi,z)| \leq \frac{((h-c)^+ + p)||f||}{1-\delta_0(x)}, \tag{12.2.45}$$

where $||f|| = \sup_{x,z} f(x|z)$.

We next obtain an estimate on $\chi_\alpha(x,z) - \chi_\alpha(x,z')$. Assume first $x > S_\alpha(z), x > S_\alpha(z')$. Then, from (12.2.41), we have

$$\begin{aligned} \chi_\alpha(x,z) - \chi_\alpha(x,z') &= g_\alpha(x,z) - g_\alpha(x,z') \\ &\quad + \alpha\int_0^x \chi_\alpha(x-\zeta,\zeta)(f(\zeta|z) - f(\zeta|z'))d\zeta. \end{aligned}$$

From the estimate (12.2.43) and the first assumption (12.2.29), we deduce

$$\begin{aligned} |\chi_\alpha(x,z) - \chi_\alpha(x,z')| \quad &\leq \quad \left(p + (h-c)^+ + \frac{\max(h,p)}{1-\delta_0(x)}\right)\delta|z-z'| \\ &\qquad x > S_\alpha(z), x > S_\alpha(z') \end{aligned}$$

Assume now, to fix ideas that $S_\alpha(z') > x > S_\alpha(z)$. Then $\chi_\alpha(x,z') = 0$ and

$$0 = g_\alpha(S_\alpha(z),z) + \alpha E[\chi_\alpha(S_\alpha(z) - z_2, z_2)\mathbb{1}_{S_\alpha(z)>z_2}|z_1 = z].$$

Therefore

$$\begin{aligned}
&\chi_\alpha(x,z)-\chi_\alpha(x,z') \\
&\quad = g_\alpha(x,z)-g_\alpha(S_\alpha(z),z)+\alpha E[\chi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1=z] \\
&\qquad -\alpha E[\chi_\alpha(S_\alpha(z)-z_2,z_2)\mathbb{1}_{S_\alpha(z)>z_2}|z_1=z] \\
&\quad = (p+\alpha(h-c))(F(x|z)-F(S_\alpha(z)|z)) \\
&\qquad +\alpha E[\chi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2>S_\alpha(z)}|z_1=z] \\
&\qquad +\alpha E[(\chi_\alpha(x-z_2,z_2)-\chi_\alpha(S_\alpha(z)-z_2,z_2))\mathbb{1}_{S_\alpha(z)>z_2}|z_1=z].
\end{aligned}$$

It follows

$$\begin{aligned}
&|\chi_\alpha(x,z)-\chi_\alpha(x,z')| \\
&\quad \leq \left[p+(h-c)^+ + \frac{\max(h,p)}{1-\delta_0(x)} + \frac{(h-c)^+ + p}{1-\delta_0(x)}\right]||f||(x-S_\alpha(z)).
\end{aligned}$$

Finally, we can state the estimate

$$|\chi_\alpha(x,z)-\chi_\alpha(x,z')| \leq \frac{C(m_0)}{1-\delta_0(x)}\delta|z-z'|, \tag{12.2.46}$$

where $C(m_0)$ depends only of constants and of m_0. Therefore, considering the gradient of χ_α in both variables, we have obtained the estimate

$$|D\chi_\alpha(x,z)| \leq \frac{C}{1-\delta_0(x)}. \tag{12.2.47}$$

From this estimate, we can assert that, for a subsequence (still denoted α)

$$\sup_{\substack{x\leq M\\ z\leq N}} |\chi_\alpha(x,z)-\chi(x,z)| \to 0, \text{as } \alpha\to 0, \forall M,N. \tag{12.2.48}$$

Therefore also

$$\begin{aligned}
&\sup_{\substack{x\leq M\\ z}} |\alpha E[\chi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1=z]| \\
&\quad \leq \sup_{\substack{x\leq M\\ z}} |E[\chi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1=z]| \\
&\qquad +(1-\alpha)\frac{\max(h,p)}{1-\delta_0(M)},
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\substack{x\leq M\\ z}} |E[\chi_\alpha(x-z_2,z_2)\mathbb{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbb{1}_{x>z_2}|z_1=z]| \\
&\quad \leq \sup_{\substack{x\leq M\\ z\leq N}} |\chi_\alpha(x,z)-\chi(x,z)| + 2\frac{\max(h,p)}{1-\delta_0(M)}\frac{c_1}{N},
\end{aligned}$$

from which we deduce easily that

$$\sup_{\substack{x \leq M \\ z}} |\alpha E[\chi_\alpha(x - z_2, z_2)\mathbb{1}_{x>z_2} - E[\chi(x - z_2, z_2)\mathbb{1}_{x>z_2}|z_1 = z]| \to 0, \forall M$$

From (12.2.41), it follows that

$$\chi(x, z) = g(x, z) + E[\chi(x - z_2, z_2)\mathbb{1}_{x>z_2}|z_1 = z],\ \forall x > S(z),$$

where

$$g(x, z) = h - (p + h - c)\bar{F}(x|z). \tag{12.2.49}$$

Also χ(x,z)=0, if $x < S(z)$. Moreover, from

$$0 = g_\alpha(S_\alpha(z), z) + \alpha E[\chi_\alpha(S_\alpha(z) - z_2, z_2)\mathbb{1}_{S_\alpha(z)>z_2}|z_1 = z]$$

and

$$\begin{aligned}
&|\alpha E[\chi_\alpha(S_\alpha(z)-z_2, z_2)\mathbb{1}_{S_\alpha(z)>z_2}|z_1=z]-E[\chi(S_\alpha(z)-z_2, z_2)\mathbb{1}_{S_\alpha(z)>z_2}|z_1=z]| \\
&\quad\leq \sup_{\substack{x \leq m_0 \\ z}} |\alpha E[\chi_\alpha(x - z_2, z_2)\mathbb{1}_{x>z_2} - E[\chi(x - z_2, z_2)\mathbb{1}_{x>z_2}|z_1 = z]|
\end{aligned}$$

$$\begin{aligned}
&|E[\chi(S_\alpha(z) - z_2, z_2)\mathbb{1}_{S_\alpha(z)>z_2}|z_1 = z] - E[\chi(S(z) - z_2, z_2)\mathbb{1}_{S(z)>z_2}|z_1 = z]| \\
&\quad\leq \frac{((h-c)^+ + p)||f||}{1 - \delta_0(m_0)}|S_\alpha(z) - S(z)| + \frac{2\max(h, p)}{1 - \delta_0(m_0)}|F(S_\alpha(z)|z) - F(S(z)|z)|
\end{aligned}$$

we obtain easily

$$0 = g(S(z), z) + E[\chi((S(z) - z_2)^+, z_2)|z_1 = z],$$

and the function $\chi(x, z)$ is continuous in x.

Let us next set

$$\Gamma_\alpha(z) = u_\alpha(0, z);$$
$$G_\alpha(z) = E[(S_\alpha(z) - z_2)^-|z_1 = z],$$

then from the first equation (12.2.26) one can check

$$\begin{aligned}
\Gamma_\alpha(z) &= cS_\alpha(z) + pG_\alpha(z) + \alpha E[u_\alpha((S_\alpha(z) - z_2)^+, z_2)|z_1 = z] \\
&= \Psi_\alpha(z) + \alpha E[\Gamma_\alpha(z_2)|z_1 = z],
\end{aligned}$$

with

$$\Psi_\alpha(z) = cS_\alpha(z) + pG_\alpha(z) + \alpha E[\int_0^{(S_\alpha(z)-z_2)^+} (h - c + \chi_\alpha(\xi, z_2))d\xi|z_1 = z,$$

and

$$0 \leq \Psi_\alpha(z) \leq \left[\max(h, c) + \frac{\max(h, p)}{1 - \delta(m_0)}\right] m_0 + pc_1.$$

This estimate also holds for the limit

$$\Psi(z) \quad = \quad cS(z) + pG(z) + E\left[\int_0^{(S(z)-z_2)^+} (h - c + \chi(\xi, z_2))d\xi|z_1 = z\right]$$

with

$$G(z) = E[(S(z) - z_2)^-|z_1 = z].$$

Define

$$\rho \quad = \quad \int \Psi(z)\varpi(z)dz.$$

Consider now the equation

$$\Gamma(z) + \rho = \Psi(z) + E[\Gamma(z_2)|z_1 = z], \qquad \int \Gamma(z)\varpi(z)dz = 0.$$

From ergodic theory, see Chapter 3, Theorem 3.3 we can assert that

$$\sup_z |\Gamma(z)| \leq \sup_z |\Psi(z) - \rho| \frac{3-\beta}{1-\beta},$$

where $0 < \beta < 1$, depends only on the Markov chain. Similarly, if we set

$$\rho_\alpha = \int \Psi_\alpha(z)\varpi(z)dz, \qquad \tilde{\Gamma}_\alpha(z) = \Gamma_\alpha(z) - \frac{\rho_\alpha}{1-\alpha},$$

we can write

$$\tilde{\Gamma}_\alpha(z) + \rho_\alpha = \Psi_\alpha(z) + \alpha E[\tilde{\Gamma}_\alpha(z_2)|z_1 = z],$$

we can also assert that

$$\begin{aligned} \sup_z |\tilde{\Gamma}_\alpha(z)| \quad &\leq \quad \sup_z |\Psi_\alpha(z) - \rho_\alpha| \frac{3-\beta}{1-\beta} \\ &\leq \quad 2\left[\left(\max(h,c) + \frac{\max(h,p)}{1-\delta(m_0)}\right) m_0 + pc_1\right] \frac{3-\beta}{1-\beta} \end{aligned}$$

Moreover

$$\tilde{\Gamma}_\alpha(z) - \tilde{\Gamma}_\alpha(z') = \Psi_\alpha(z) - \Psi_\alpha(z') + \alpha \int \tilde{\Gamma}_\alpha(\zeta)(f(\zeta|z) - f(\zeta|z'))d\zeta.$$

Using properties (12.2.40), (12.2.43) and the assumption (12.2.29), we can check that

$$|\Psi_\alpha(z) - \Psi_\alpha(z')| \leq C(m_0)\delta|z - z'|,$$

and thus also

$$|\tilde{\Gamma}_\alpha(z) - \tilde{\Gamma}_\alpha(z')| \leq C_1(m_0)\delta|z - z'|.$$

Therefore the functions $\tilde{\Gamma}_\alpha(z)$ are uniformly Lipschitz continuous and bounded. It follows that for a subsequence

we obtain

$$\sup_{0 \leq z \leq N} \Gamma_\alpha \left|(z) - \frac{\rho_\alpha}{1-\alpha} - \Gamma(z)\right| \to 0, \ \forall M \tag{12.2.50}$$

Therefore, also

$$\sup_{\substack{0 \leq x \leq M \\ 0 \leq z \leq N}} \left|u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)\right| \to 0, \forall x, \forall M, N \tag{12.2.51}$$

with

$$u(x,z) = (h-c)x + + \int_0^x \chi(\xi,z)d\xi + \Gamma(z). \tag{12.2.52}$$

We deduce

$$u(x,z) = (h-c)x + \Gamma(z), \forall x \leq S(z). \tag{12.2.53}$$

From (12.2.50), (12.2.51) it is clear that

$$\Gamma(z) = u(0, z). \tag{12.2.54}$$

However

$$\begin{aligned} &E[u((S(z) - z_2)^+, z_2 | z_1 = z] \\ &\quad = E[u(0, z_2)|z_1 = z] + E\left[\int_0^{(S(z)-z_2)^+} (h - c + \chi(\xi, z_2))d\xi | z_1 = z\right], \end{aligned}$$

hence

$$\Gamma(z) = -\rho + cS(z) + pG(z) + E[u((S(z) - z_2)^+, z_2 | z_1 = z],$$

and thus the first relation (12.2.32) is proven.

Consider now the situation with $x \geq S(z)$. Define the function

$$\tilde{u}(x, z) = hx + pE[(x - z_2)^- | z_1 = z] + E[u((x - z_2)^+, z_2)|z_1 = z].$$

We obtain

$$\begin{aligned} \tilde{u}'(x, z) &= h - p\bar{F}(x|z) + E[u'(x - z_2, z_2)1\!\!1_{x>z_2} | z_1 = z] \\ &= h - c + \chi(x, z), \quad x \geq S(z). \end{aligned}$$

Also

$$\begin{aligned} \tilde{u}(S(z), z) &= hS(z) + pE[(S(z) - z_2)^- | z_1 = z] + \Gamma(z) \\ &= u(S(z), z) + \rho. \end{aligned}$$

From these two relations we get $\tilde{u}(x, z) = u(x, z) + \rho, \forall x \geq S(z)$.

This concludes the second part of (12.2.32). Note also that

$$u_\alpha(x, z) - u_\alpha(x, z') = \Gamma_\alpha(z) - \Gamma_\alpha(z') + \int_0^x (\chi_\alpha(\xi, z) - \chi_\alpha(\xi, z')d\xi.$$

Using already proven estimates we obtain

$$|u_\alpha(x, z) - u_\alpha(x, z')| \leq \left[C_0(m_0) + \frac{C_1(m_0)x}{1 - \delta_0(x)}\right] \delta|z - z'|$$

The limit $u(x, z)$ satisfies all the estimates (12.2.34), (12.2.35).

To prove (12.2.36), we first check that

$$u_\alpha(x, z) \leq l(x, z, v) + \alpha E[u_\alpha((x + v - z_2)^+, z_2)|z_1 = z], \forall x, z, v$$

Therefore, it easily follows that

$$u(x, z) + \rho \leq l(x, z, v) + E[u((x + v - z_2)^+, z_2)|z_1 = z], \forall x, z, v$$

However (12.2.36) can be read as

$$u(x, z) + \rho = l(x, z, \hat{v}(x, z)) + E[u((x + \hat{v}(x, z) - z_2)^+, z_2)|z_1 = z], \forall x, z$$

where

$$\hat{v}(x, z) = \begin{cases} S(z) - x, & \text{if } x \leq S(z) \\ 0, & \text{if } x \geq S(z) \end{cases}$$

Combining we get the equation (12.2.36).

Let us prove uniqueness, for $S(z)$ given. We first prove that $\chi(x, z)$ is uniquely defined. To prove this it is sufficient to prove that if

$$\chi(x, z) = \begin{cases} E[\chi(x - z_2, z_2)1\!\!1_{x>z_2} | z_1 = z], & \forall x > S(z) \\ 0, & \forall x \leq S(z) \end{cases}$$

and

$$(1-\delta_0(x)) \sup_{\substack{0 \le \xi \le x \\ z}} |\chi(\xi, z)| < \infty,$$

then $\chi(x,z) = 0$. This is clear. The function $\Psi(z)$ is thus uniquely defined. It follows that the pair ρ, $\Gamma(z) = u(0,z)$ is also uniquely defined, with the condition $\int \Gamma(z)\varpi(z)dz = 0$. Therefore $u(x,z)$ is also uniquely defined. The proof of the theorem has been completed. □

Example 12.1. Consider the situation of independent demands, then $f(x|z) = f(x)$. In that case $\varpi(x) = f(x)$. Then $S(z) = S$, and

$$\rho = (p+h-c)E(S-D)^+ - (p-c)S + p\bar{D}$$

and we recover the formula (6.2.4) of Chapter 6. We see also that $\Psi(z) = \rho$, and thus $\Gamma(z) = 0$.

Consider next the situation

$$f(x|z) = \lambda(z)\exp -\lambda(z)x,$$

with the assumption $0 < \lambda_0 \le \lambda(z) \le \lambda_1$. We also assume that $\lambda(z)$ is Lipschitz continuous. Then all assumptions of Theorem 12.3 are satisfied.

We turn to the interpretation of the number ρ. Consider the feedback $\hat{v}(x,z)$ associated with the base stock $S(z)$. Define the controlled process

$$\hat{y}_{n+1} = (\hat{y}_n + \hat{v}_n - z_{n+1})^+,\ \hat{v}_n = \hat{v}(\hat{y}_n, z_n),$$

with $\hat{y}_1 = x,\ z_1 = z$. We define the policy $\hat{V} = (\hat{v}_1, \cdots, \hat{v}_n, \cdots)$. We consider the averaged cost

$$J^n_{x,z}(\hat{V}) = \frac{\sum_{j=1}^n El(\hat{y}_j, z_j, \hat{v}_j)}{n}.$$

Similarly, for any policy $V = (v_1, \cdots, v_n \cdots)$ we define the averaged cost

$$J^n_{x,z}(V) = \frac{\sum_{j=1}^n El(y_j, z_j, v_j)}{n},$$

with

$$y_{n+1} = (y_n + v_n - z_{n+1})^+, \quad y_1 = x,\ z_1 = z.$$

We state the

Proposition 12.1. *We have the property*

$$\rho = \lim_{n\to\infty} J^n_{x,z}(\hat{V}). \tag{12.2.55}$$

Furthermore, consider the set of policies

$$\mathcal{U} = \{V \mid E|u(y_n, z_n)| \le C_x\},$$

then we have

$$\rho = \inf_{V\in\mathcal{U}} \lim_{n\to\infty} J^n_{x,z}(V). \tag{12.2.56}$$

PROOF. We first notice that

$$\hat{y}_n \leq \max(x, m_0).$$

Therefore, from the estimate (12.2.33), we get

$$|u(\hat{y}_n, z_n)| \leq C_0 + \frac{C_1 \max(x, m_0)}{1 - \delta_0(\max(x, m_0))}.$$

Therefore $\hat{V}$ belongs to $\mathcal{U}$. From (12.2.36) we can write

$$u(\hat{y}_n, z_n) + \rho = l(\hat{y}_n, z_n, \hat{v}_n) + E[u(\hat{y}_{n+1}, z_{n+1})|\hat{y}_n, z_n].$$

Taking the expectation and adding up we obtain

$$\rho = J^n_{x,z}(\hat{V}) + \frac{Eu(\hat{y}_{n+1}, z_{n+1}) - u(x, z)}{n},$$

and thus the property (12.2.55) follows immediately. Similarly, for any policy we can write

$$\rho \leq J^n_{x,z}(V) + \frac{Eu(y_{n+1}, z_{n+1}) - u(x, z)}{n}.$$

Therefore, if $V \in \mathcal{U}$, we have $\rho \leq J^n_{x,z}(V)$. This implies (12.2.56). The proof has been completed. □

Remark 12.2. We cannot state that the process $\hat{y}_n, z_n$ is ergodic. We cannot indeed apply the property (3.5.5) of Theorem 3.3, Chapter 3. Consequently, we cannot give an interpretation of the function $u(x, z)$ itself.

12.3. BACKLOG AND NO SET UP COST

12.3.1. THE MODEL. We consider here the situation where backlog is permitted. This is the equivalent of Chapter 5, section 5.2. The Markov chain representing the demand is unchanged. The transition probability is $f(\varsigma|z)$. We recall that

(12.3.1) $f(\zeta|z)$ is uniformly continuous in both variables and bounded

$$\int_0^{+\infty} \zeta f(\zeta|z) d\zeta \leq c_0 z + c_1. \tag{12.3.2}$$

We define the filtration

$$\mathcal{F}^n = \sigma(z_1, \cdots, z_n).$$

A control policy, denoted by V, is a sequence of random variables v_n such that v_n is $\mathcal{F}^n$ measurable. When $z_1 = z$, then v_1 is deterministic. Also, we assume as usual that $v_n \geq 0$. We next define the inventory as the sequence,

$$y_{n+1} = y_n + v_n - z_{n+1}, \quad y_1 = x. \tag{12.3.3}$$

The process y_n is adapted to the filtration $\mathcal{F}^n$. The joint process y_n, z_n is also a Markov chain.

We can write, for a test function $\varphi(x, z)$ (bounded continuous on $R \times R^+$)

$$\begin{aligned} E[\varphi(y_2, z_2)|y_1 = x, z_1 = z, v_1 = v] &= E[\varphi(x + v - z_2, z_2)|z_1 = z] \\ &= \int_0^{\infty} \varphi(x + v - \zeta, \zeta) f(\zeta|z) d\varsigma \end{aligned}$$

This defines a Markov chain to which is associated the operator

$$\Phi^v\varphi(x,z) = \int_0^\infty \varphi(x+v-\zeta,\zeta)f(\zeta|z)d\varsigma, \tag{12.3.4}$$

hence the transition probability is given by

$$\pi(x,z,v;d\xi,d\zeta) = \delta(\xi-(x+v-\zeta))\otimes f(\zeta|z)d\zeta. \tag{12.3.5}$$

We next define the function

$$l(x,v) = cv + hx^+ + px^-, \tag{12.3.6}$$

which will model the one period cost. The cost function is then

$$J_{x,z}(V) = E\sum_{n=1}^\infty \alpha^{n-1}l(y_n,v_n). \tag{12.3.7}$$

We are interested in the value function

$$u(x,z) = \inf_V J_{x,z}(V). \tag{12.3.8}$$

We notice again that the properties (4.3.1), (4.3.2) are satisfied. To proceed, we need to specify a ceiling function. It corresponds to a control identically 0. Set $l_v(x) = l(x,v)$. Note the inequalities

$$\Phi^0x^+(x,z) \le x^+, \quad \Phi^0x^-(x,z) \le x^- + c_0z + c_1.$$

It is easy to check that

$$(\Phi^0)^n x^+(x,z) \quad \le \quad x^+, \tag{12.3.9}$$

$$(\Phi^0)^n x^-(x,z)(x,z) \le x^- + c_0z\frac{1-c_0^n}{1-c_0} + (n-c_0(n+1)+c_0^{n+1})\frac{c_1}{(1-c_0)^2}. \tag{12.3.10}$$

Consider then the function

$$w_0(x,z) = \sum_{n=1}^\infty \alpha^{n-1}(\Phi^0)^{n-1}l_0(x) \tag{12.3.11}$$

Lemma 12.2. *We assume that*

$$\alpha c_0 < 1, \tag{12.3.12}$$

then the series $w_0(x,z) < \infty$, and more precisely

$$w_0(x,z) \le \frac{hx^+ + px^-}{1-\alpha} + \frac{pc_0z\alpha}{(1-c_0\alpha)(1-\alpha)} + \frac{pc_1}{(1-c_0)^2}\left(\frac{\alpha-c_0}{(1-\alpha)^2} + \frac{c_0}{1-\alpha c_0}\right). \tag{12.3.13}$$

PROOF. It is an immediate consequence of formulas (13.2.10) and (12.3.9). $\square$

We next write the Bellman equation

$$u(x,z) = \inf_{v\ge 0}[l(x,v) + \alpha\Phi^v u(x,z)]. \tag{12.3.14}$$

We state the following

Theorem 12.4. *We assume* (13.2.1), (12.3.2), (12.3.6), (13.2.11). *Then there exists one and only one solution of equation* (13.2.15)*, such that $0 \le u \le w_0$. It is continuous and coincides with the value function* (12.3.8)*. There exists an optimal feedback $\hat{v}(x,z)$ and an optimal control policy $\hat{V}$.*

PROOF. We first notice that conditions of Theorem 4.4 and 4.3 are satisfied. Hence the set of solutions of equation (13.2.15) is not empty. It has a minimum and a maximum solution. The minimum solution is l.s.c. and the maximum solution is u.s.c. The minimum solution coincides with the value function. There exists an optimal feedback $\hat{v}(x,z)$ and an optimal control policy $\hat{V}$. To prove uniqueness, we have to prove that the minimum and maximum solutions coincide. We begin by proving a bound on $\hat{v}(x,z)$. From (13.2.15), considering $\underline{u}$, the minimum solution, we can state that

$$\underline{u}(x,z) \geq \frac{hx^+}{1-\alpha} + px^- - \frac{\alpha h c_0 z}{(1-\alpha)(1-\alpha c_0)} - \frac{\alpha h c_1}{(1-\alpha)^2}, \tag{12.3.15}$$

which follows from $l(x,v) \geq hx^+ + px^-$, and

$$\Phi^v \varphi(x,z) \geq \Phi^0 \varphi(x,z), \forall v \geq 0, \forall \varphi \text{ increasing in } x, \tag{12.3.16}$$

which implies

$$\begin{aligned}
&\alpha E[\underline{u}(x+v-z_2,z_2)|z_1=z] \\
&\geq \frac{\alpha h}{1-\alpha} E[(x-z_2)^+|z_1=z] - \frac{\alpha^2 h c_0(c_0 z + c_1)}{(1-\alpha)(1-\alpha c_0)} - \frac{\alpha^2 h c_1}{(1-\alpha)^2} \\
&\geq \frac{\alpha h}{1-\alpha}(x^+ - c_0 z - c_1) - \frac{\alpha^2 h c_0(c_0 z + c_1)}{(1-\alpha)(1-\alpha c_0)} - \frac{\alpha^2 h c_1}{(1-\alpha)^2}
\end{aligned}$$

and (13.2.40) follows, using again (13.2.15).

Therefore also

$$l(x,v) + \alpha \Phi^v \underline{u}(x,z) \geq cv + \frac{hx^+}{1-\alpha} + px^- - \frac{\alpha h c_0 z}{(1-\alpha)(1-\alpha c_0)} - \frac{\alpha h c_1}{(1-\alpha)^2}. \tag{12.3.17}$$

We next introduce the function $w(x,z)$ solution of

$$w(x,z) = hx^+ + (c+p)x^- + \alpha E[w(x^+ - z_2, z_2)|z_1=z], \tag{12.3.18}$$

which corresponds to the feedback $v(x,z) = x^-$. We note that $\underline{u}(x,z) \leq w(x,z)$.

We can assert that

$$w(x,z) \leq (c+p)x^- + \frac{hx^+}{1-\alpha} + L_0 z + L_1, \tag{12.3.19}$$

with

$$L_0 = \frac{\alpha(c+p)c_0}{1-\alpha c_0}, \quad L_1 = \frac{\alpha c_1(c+p)}{(1-\alpha)(1-\alpha c_0)}$$

Therefore, in minimizing in v, we can bound from above the range of v. More precisely, we get

(12.3.20)

$$\hat{v}(x,z) \leq x^- + \frac{\alpha c_0 z}{c(1-c_0\alpha)}\left(c+p+\frac{h}{1-\alpha}\right) + \frac{\alpha c_1}{c(1-\alpha)}\left(\frac{h}{1-\alpha} + \frac{c+p}{1-c_0\alpha}\right).$$

We consider the optimal trajectory $\hat{y}_n, z_n$ obtained from the optimal feedback, namely

$$\hat{y}_{n+1} = \hat{y}_n + \hat{v}_n - z_{n+1}, \; \hat{v}_n = \hat{v}(\hat{y}_n, z_n)$$

with $\hat{y}_1 = x, z_1 = z$.

To prove uniqueness, it is sufficient to show that

$$\alpha^n E|\hat{y}_{n+1}| \to 0, \text{ as } n \to \infty. \tag{12.3.21}$$

We have

$$\begin{aligned}\hat{y}_{n+1} &\leq \hat{y}_n + \hat{v}_n \\ &\leq \hat{y}_n^+ + \frac{\alpha c_0 z_n}{c(1-c_0\alpha)}\left(c+p+\frac{h}{1-\alpha}\right) + \frac{\alpha c_1}{c(1-\alpha)}\left(\frac{h}{1-\alpha}+\frac{c+p}{1-c_0\alpha}\right)\end{aligned}$$

Therefore

$$E\hat{y}_{n+1}^+ \leq E\hat{y}_n^+ + \frac{\alpha\left(c_0^n z + c_1\frac{c_0-c_0^n}{1-c_0}\right)}{c(1-c_0\alpha)}\left(c+p+\frac{h}{1-\alpha}\right)$$
$$+\frac{\alpha c_1}{c(1-\alpha)}\left(\frac{h}{1-\alpha}+\frac{c+p}{1-c_0\alpha}\right)$$

and we deduce the estimate

$$E\hat{y}_{n+1}^+ \leq x^+ + \frac{\alpha c_0\left(z\frac{1-c_0^n}{1-c_0}+c_1\frac{n}{1-c_0}-c_1\frac{1-c_0^n}{(1-c_0)^2}\right)}{c(1-c_0\alpha)}\left(c+p+\frac{h}{1-\alpha}\right)$$
$$+\frac{n\alpha c_1}{c(1-\alpha)}\left(\frac{h}{1-\alpha}+\frac{c+p}{1-c_0\alpha}\right),$$

and thus $\alpha^n E\hat{y}_{n+1}^+ \to 0$, as $n \to +\infty$. Next

$$\hat{y}_{n+1} \geq \hat{y}_n - z_{n+1},$$

and it follows easily

$$E\hat{y}_{n+1}^- \leq x^- + c_0 z\frac{1-c_0^n}{1-c_0} + \frac{c_1}{1-c_0}\left(n - c_0\frac{1-c_0^n}{1-c_0}\right),$$

and we obtain $\alpha^n E\hat{y}_{n+1}^- \to 0$, as $n \to +\infty$. This completes the proof. □

12.3.2. BASE-STOCK POLICY. We want now to check that the optimal feedback $\hat{v}(x,z)$ can be obtained by a base stock policy, with a base stock depending on the value of z. We have the

Theorem 12.5. *We make the assumptions of Theorem 12.4 and* $p\alpha > c(1-\alpha)$, $h+p-c>0$. *Assume also*

(12.3.22) $$f(x|z) \geq a_0(M) > 0, \ \forall x, z \leq M.$$

Then the function $u(x,z)$ *is convex and* C^1 *in the argument* x, *except for* $x=0$, *where there exists a left and right derivative. Moreover the optimal feedback is given by*

(12.3.23) $$\hat{v}(x,z) = \begin{cases} S(z)-x, & \text{if } x \leq S(z) \\ 0, & \text{if } x \geq S(z)\end{cases}$$

where $S(z) > 0$ *is uniformly continuous.*

Proof. We consider the increasing process

$$u_{n+1}(x,z) = \inf_{v\geq 0}[l(x,v)+\alpha\Phi^v u_n(x,z)], \qquad u_0(x)=0$$

which is also written as

$$u_{n+1}(x,z) = hx^+ + px^- - cx + \inf_{\eta\geq x}\{c\eta + \alpha E[u_n(\eta - z_2, z_2)|z_1 = z]\}$$

We show recursively that the sequence $u_n(x,z)$ is convex and C^1 in x, except for $x=0$. We first have

$$u_{n+1}(x,z) = hx^+ + px^- + \alpha E[u_n(x-z_2,z_2)|z_1=z] \tag{12.3.24}$$

for $0 \le n \le n_0$, with

$$\frac{\alpha p}{1-\alpha}(1-\alpha^{n_0+1}) > c > \frac{\alpha p}{1-\alpha}(1-\alpha^{n_0}).$$

Indeed, considering the sequence defined by (13.2.24), we have

$$u_n'(x,z) = -p\frac{1-\alpha^n}{1-\alpha}, \forall n \le n_0+1, \forall x < 0$$

Therefore $c - \alpha p \dfrac{1-\alpha^n}{1-\alpha} > 0, \forall n \le n_0$ and the function $c\eta + \alpha E[u_n(\eta - z_2, z_2)|z_1 = z]$ is monotone increasing, $\forall n \le n_0$.

Define

$$G_n(x,z) \quad = \quad cx + \alpha E[u_n(x-z_2,z_2)|z_1=z],$$

then the function $G_n(x,z)$ is monotone increasing for $n \le n_0$, and for $n = n_0+1$ attains its minimum in $S_n(z) > 0$, which can be uniquely defined by taking the smallest minimum. It is C^1 and convex (the derivative is continuous in 0, by the regularizing effect of the mathematical expectation).

We have then, for $n \le n_0+1$

$$u_{n+1}(x,z) = \begin{cases} hx^+ + px^- - cx + G_n(S_n(z),z) & \text{if } x \le S_n(z) \\ hx^+ + px^- - cx + G_n(x,z) & \text{if } x \ge S_n(z) \end{cases}$$

with $S_n(z) = -\infty, \forall n \le n_0$. Hence $u_n(x,z)$ is C^1, except for $x=0$, and convex, for $n \le n_0+2$. By recurrence, this property is valid for any n.

It follows that the limit $u(x,z)$ is convex in x. Clearly $u(x,z) \to +\infty$, as $|x| \to +\infty$. Also $S_n(z) > 0, \forall n \ge n_0+1$. Moreover

$$u_n'(x,z) = -p-c, \forall x < 0, \forall n \ge n_0+2$$

and $u_n'(x,z) \le \dfrac{h}{1-\alpha}$. Therefore the limit $u(x,z)$ is Lipschitz continuous in x, and

$$u'(x,z) = -p-c, \forall x<0, \qquad u'(x,z) \le \frac{h}{1-\alpha}.$$

Also

$$-p-c \le u'(x,z) \le \frac{h}{1-\alpha}. \tag{12.3.25}$$

The function

$$G(x,z) = cx + \alpha E[u(x-z_2,z_2)|z_1=z],$$

is convex and $\to +\infty$ as $x \to +\infty$. For $x<0$, we have $G'(x,z) = c(1-\alpha) - \alpha p < 0$. Therefore the minimum of $G(x,z)$ is attained in $S(z) > 0$, which can be defined in a unique way, by taking the smallest minimum.

Then, from convexity

$$u(x,z) = \begin{cases} hx^+ + px^- - cx + G(S(z),z) & \text{if } x \le S(z) \\ hx^+ + px^- - cx + G(x,z) & \text{if } x \ge S(z) \end{cases}$$

So in fact, we can write a better estimate than the left hand-side of (12.3.25)

$$u'(x,z) \geq h - c, \quad \text{if } x \geq 0. \tag{12.3.26}$$

Therefore also

$$G'(x,z) \geq c(1-\alpha) + \alpha h - \alpha(p+h)\bar{F}(x|z),$$

and since $S(z)$ is the minimum, we obtain

$$\bar{F}(S(z)|z) \geq \frac{c(1-\alpha)+\alpha h}{\alpha(p+h)}$$

which implies

$$0 < S(z) < \frac{(c_0 z + c_1)\alpha(p+h)}{c(1-\alpha)+\alpha h}. \tag{12.3.27}$$

Also the function $S(z)$ is continuous and thus the feedback $\hat{v}(x,z)$ is continuous.

If we define

$$\chi(x,z) = u'(x,z) - h\mathbb{1}_{x>0} + p\mathbb{1}_{x<0} + c,$$

as an element of B (space of measurable bounded functions on $R \times R^+$), then χ is the unique solution in B of the equation

$$\chi(x,z) = g(x+\hat{v}(x,z),z) + \alpha E[\chi(x+\hat{v}(x,z)-z_2,z_2)|z_1 = z], \; x \in R, z \in R^+$$

where

$$g(x,z) = c(1-\alpha) + \alpha h - \alpha(p+h)\bar{F}(x|z).$$

Since the function $g(x+\hat{v}(x,z),z)$ is continuous in the pair x,z, the solution $\chi(x,z)$ is also continuous.

We next show that $S(z)$ is uniformly continuous. Consider $G'(x_1,z) - G'(x_2,z)$, for $x_1 \geq x_2 \geq 0$. We have

$$\begin{aligned}
&G'(x_1,z) - G'(x_2,z)\\
&\quad = \alpha E[u'(x_1-z_2,z_2) - u'(x_2-z_2,z_2)|z_1 = z]\\
&\quad = \alpha E[u'(x_1-z_2,z_2)\mathbb{1}_{x_1-z_2>0} + u'(x_1-z_2,z_2)\mathbb{1}_{x_1-z_2\leq 0}\\
&\qquad\quad - u'(x_2-z_2,z_2)\mathbb{1}_{x_2-z_2>0} - u'(x_2-z_2,z_2)\mathbb{1}_{x_2-z_2\leq 0}|z_1 = z]\\
&\quad = \alpha E[(u'(x_1-z_2,z_2) - u'(x_2-z_2,z_2))\mathbb{1}_{x_2-z_2>0}\\
&\qquad\quad + u'(x_1-z_2,z_2)\mathbb{1}_{x_2<z_2<x_1}|z_1 = z] + \alpha E[(u'(x_1-z_2,z_2)\\
&\qquad\quad - u'(x_2-z_2,z_2))\mathbb{1}_{x_1-z_2\leq 0} - u'(x_2-z_2,z_2)\mathbb{1}_{x_2<z_2<x_1}|z_1 = z]\\
&\quad \geq \alpha(h+p)(F(x_1|z) - F(x_2|z)).
\end{aligned}$$

Therefore

$$(G'(x_1,z) - G'(x_2,z))(x_1-x_2) \geq \alpha(h+p)(F(x_1|z) - F(x_2|z))(x_1-x_2). \tag{12.3.28}$$

We can check easily that this formula holds for any pair x_1, x_2. We deduce as in Theorem 12.2, that for $z, z' < m$ and $S(z), S(z') < M_m$, with $M_m > m$, the following relation holds

$$(G'(S(z),z') - G'(S(z'),z))(S(z) - S(z')) \geq 2\alpha(h+p)a_0(M_m)(S(z) - S(z'))^2.$$

Next

$$\begin{aligned}G'(x,z) - G'(x,z') \\ &= \alpha E[u'(x-z_2,z_2)|z_1=z] - E[u'(x-z_2,z_2)|z_1=z'] \\ &= (p+c)(F(x|z)-F(x|z')) + \int_0^{x^+} u'(x-\zeta,\zeta)(f(\zeta|z)-f(\zeta|z'))d\zeta,\end{aligned}$$

hence

$$|G'(x,z)-G'(x,z')| \le Cx^+ \sup_{0<\zeta<x^+} |f(\zeta|z)-f(\zeta|z')|$$

Applying this inequality for $x = S(z)$ and $x = S(z')$ respectively, and combining estimates we deduce that $S(z)$ is uniformly continuous. This completes the proof. □

12.3.3. ERGODIC THEORY. We turn now to the case when $\alpha \to 1$. We write $u_\alpha(x,z)$ as follows

$$(12.3.29)\qquad u_\alpha(x,z) = \begin{cases} hx^+ + px^- - cx + cS_\alpha(z) \\ \qquad +\alpha E[u_\alpha(S_\alpha(z)-z_2,z_2)|z_1=z] & \text{if } x \le S_\alpha(z) \\ hx^+ + px^- \\ \qquad +\alpha E[u_\alpha(x-z_2,z_2)|z_1=z] & \text{if } x \ge S_\alpha(z)\end{cases}$$

We shall make the assumptions

$$(12.3.30)\qquad \begin{aligned} c_0 &= 0 \\ \inf_{\substack{0\le\zeta\le a \\ z}} f(\zeta|z) &\ge \gamma(a) > 0, \forall a \end{aligned}$$

$$(12.3.31)\qquad f(\zeta|z) \text{ is ergodic}$$

$$(12.3.32)\qquad \begin{aligned} \int |f(\zeta|z)-f(\zeta|z')|d\zeta &\le \delta|z-z'| \\ F(x|z) &\le \delta_0(x) < 1, \forall x\end{aligned}$$

We denote by $\varpi(z)$ the invariant probability density corresponding to the Markov chain $f(\zeta|z)$.

We state the

Theorem 12.6. *We assume* (13.2.1), (12.3.2) *with* $c_0 = 0$, (12.3.6), (12.3.22) *and* (12.3.30), (12.3.31), (12.3.32). *Then, for a subsequence (still denoted* α*) converging to* 1, *we have, for any compact* K *of* R^+

$$(12.3.33)\qquad \sup_{z\in K} |S_\alpha(z)-S(z)| \le \epsilon(\alpha,K), \quad \epsilon(\alpha,K)\to 0, \text{ as } \alpha\uparrow 1$$

$$(12.3.34)\qquad \sup_{\substack{x\le M \\ z\le N}} u_\alpha\left|(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)\right| \to 0, \forall M,N,$$

with $\rho_\alpha \to \rho$ and

$$u(x,z)+\rho = \begin{cases} hx^+ + px^- - cx + cS(z) \\ \quad +E[u(S(z)-z_2,z_2)|z_1=z] & \text{if } x \le S(z) \\ hx^+ + px^- + E[u(x-z_2,z_2)|z_1=z] & \text{if } x \ge S(z) \end{cases} \tag{12.3.35}$$

The function $u(x,z)$ satisfies the growth condition

$$\sup_{\substack{\xi \le x \\ z}} |u(\xi,z)| \le C_0 + \frac{C_1 x}{1-\delta_0(x)}. \tag{12.3.36}$$

It is C^1 in x and Lipschitz continuous in z. The following estimates hold

$$\sup_{\substack{\xi \le x \\ z}} |u_x(\xi,z)| \le \frac{C}{(1-\delta_0(x))}, \tag{12.3.37}$$

$$|u(x,z)-u(x,z')| \le \left[C_0(m_0)+\frac{C_1(m_0)x}{1-\delta_0(x)}\right]\delta|z-z'| \tag{12.3.38}$$

$$(1-\delta_0(x)) \sup_{\substack{\xi \le x \\ z}} |u_{xx}(\xi,z)| \le C,\ (1-\delta_0(x)) \tag{12.3.39}$$

$$\sup_{\substack{\xi \le x \\ z}} |u_{xz}(\xi,z)| \le C,\ a.e.$$

Given $S(z)$, the pair $u(x,z),\rho$ satisfying the above conditions and $\int u(0,z)\varpi(z)dz = 0$ is uniquely defined.

Also

$$u(x,z)+\rho = \inf_{v\ge 0}[l(x,z,v)+\Phi^v u(x,z)]. \tag{12.3.40}$$

Proof. We begin with (12.3.33). We first note that

$$0 \le S_\alpha \le \frac{c_1(p+h)}{\min(c,h)} = m_0. \tag{12.3.41}$$

Moreover, from (12.3.28) and (12.3.30) we have

$$\begin{aligned}(G'_\alpha(S_\alpha(z),z') - G'_\alpha(S_\alpha(z'),z))(S_\alpha(z)-S_\alpha(z')) \\ \ge 2\alpha(h+p)\gamma(m_0)(S_\alpha(z)-S_\alpha(z'))^2\end{aligned}$$

Now

$$\begin{aligned}&(G'_\alpha(S_\alpha(z),z') - G'_\alpha(S_\alpha(z'),z))(S_\alpha(z)-S_\alpha(z')) \\ &\quad = \alpha\int (u'_\alpha(S_\alpha(z)-\zeta,\zeta)+u'_\alpha(S_\alpha(z')-\zeta,\zeta))(f(\zeta|z')-f(\zeta|z))d\zeta\end{aligned}$$

From (13.2.25), we also have

$$\sup_{\substack{\xi \le x \\ z}} |u'_\alpha(\xi,z)| \le \frac{\max(h,p+c)}{1-\delta_0(x)}. \tag{12.3.42}$$

Therefore

$$|G'_\alpha(S_\alpha(z),z') - G'_\alpha(S_\alpha(z'),z)| \leq \frac{2\alpha\max(h,p+c)}{1-\delta_0(m_0)}\int|f(\zeta|z') - f(\zeta|z)|d\zeta$$

Collecting results, we can then state the estimate

$$|S_\alpha(z) - S_\alpha(z')| \leq \frac{\max(h,p+c)}{(h+p)\gamma(m_0)(1-\delta_0(m_0))}\int|f(\zeta|z') - f(\zeta|z)|d\zeta$$

and using the assumption (12.3.32) we get

$$|S_\alpha(z) - S_\alpha(z')| \leq \frac{\delta\max(h,p+c)}{(h+p)\gamma(m_0)(1-\delta_0(m_0))}|z'-z|. \tag{12.3.43}$$

So the sequence $S_\alpha(z)$ is uniformly Lipschitz continuous, from which the property (12.3.33) follows.

Consider $\chi_\alpha(x,z) = u'_\alpha(x,z) - h\mathbb{1}_{x>0} + p\mathbb{1}_{x<0} + c$. We have

$$\begin{aligned}\chi_\alpha(x,z) &= g_\alpha(x,z) + \alpha E[\chi_\alpha(x-z_2,z_2)|z_1=z], \text{ if } x \geq S_\alpha(z)\\ &= 0 \quad \text{if} \quad x \leq S_\alpha(z)\end{aligned} \tag{12.3.44}$$

with

$$g_\alpha(x,z) = c + \alpha(h-c) - \alpha(h+p)\bar{F}(x|z), \tag{12.3.45}$$

we deduce easily that

$$\sup_{\substack{\xi \leq x \\ z}} |\chi_\alpha(x,z)| \leq \frac{\max(h,c)}{1-\delta_0(x)}. \tag{12.3.46}$$

Consider next $\psi_\alpha(x,z) = \chi'_\alpha(x,z)$. We obtain

$$\psi_\alpha(x,z) = \begin{cases}\alpha(h+p)f(x|z) + \alpha E[\psi_\alpha(x-z_2,z_2)|z_1=z], & \text{if } x > S_\alpha(z)\\ 0, & \text{if} \quad x < S_\alpha(z)\end{cases} \tag{12.3.47}$$

This function is not continuous but it is bounded (not uniformly in α), and we can assert that

$$\sup_{\substack{\xi \leq x \\ z}} |\psi_\alpha(\xi,z)| \leq \frac{(h+p)||f||}{1-\delta_0(x)}. \tag{12.3.48}$$

Next, consider $\chi_\alpha(x,z) - \chi_\alpha(x,z')$. For $x > S_\alpha(z)$, $x > S_\alpha(z')$, we check easily that

$$|\chi_\alpha(x,z) - \chi_\alpha(x,z')| \leq \left[h+p+\frac{\max(h,c)}{1-\delta_0(x)}\right]\delta|z-z'|.$$

Next operating as in Theorem 12.3 we can state that for $S_\alpha(z') > x > S_\alpha(z)$

$$|\chi_\alpha(x,z) - \chi_\alpha(x,z')| \leq h\left[+p+\frac{\max(h,c)}{1-\delta_0(x)} + \frac{h+p}{1-\delta_0(x)}\right]||f||(x-S_\alpha(z)).$$

Combining estimates we obtain

$$|\chi_\alpha(x,z) - \chi_\alpha(x,z')| \leq \frac{C(m_0)\delta}{1-\delta_0(x)}|z-z'|. \tag{12.3.49}$$

So considering the gradient of χ in both variables, we have

$$\sup_{\substack{\xi \leq x \\ z}} |D\chi_\alpha(x,z)| \leq \frac{C}{1-\delta_0(x)}.$$

From this estimate and using the fact that $\chi_\alpha(x,z) = 0, \forall x < 0$, we can assert that, for a subsequence (still denoted α)

$$\sup_{\substack{x \leq M \\ z \leq N}} |\chi_\alpha(x,z) - \chi(x,z)| \to 0, \text{as } \alpha \to 0, \forall M, N. \tag{12.3.50}$$

We operate as in Theorem 12.3 to show that

$$\sup_{\substack{x \leq M \\ z}} |\alpha E[\chi_\alpha(x - z_2, z_2)|z_1 = z] - E[\chi(x - z_2, z_2)|z_1 = z]| \to 0, \forall M$$

From (13.2.34), it follows that

$$\chi(x,z) = g(x,z) + E[\chi(x - z_2, z_2)|z_1 = z], \ \forall x > S(z),$$

where

$$g(x,z) = h - (p+h)\bar{F}(x|z). \tag{12.3.51}$$

Also χ(x,z)=0, if $x < S(z)$. Moreover, from

$$0 = g_\alpha(S_\alpha(z), z) + \alpha E[\chi_\alpha(S_\alpha(z) - z_2, z_2)|z_1 = z],$$

and

$$\sup_{z \leq N} |\alpha E[\chi_\alpha(S_\alpha(z) - z_2, z_2)|z_1 \quad = \quad z] - E[\chi(S(z) - z_2, z_2)1\!\!1_{S(z) > z_2}|z_1 = z]| \to 0$$

we obtain easily

$$0 = g(S(z), z) + E[\chi((S(z) - z_2)^+, z_2)|z_1 = z],$$

and the function $\chi(x,z)$ is continuous in x.

We set

$$\Gamma_\alpha(z) = E[u_\alpha(0, z_2)|z_1 = z],$$

then from the first equation (13.2.25) one can check

$$\Gamma_\alpha(z) = \Psi_\alpha(z) + \alpha E[\Gamma_\alpha(z_2)|z_1 = z],$$

with

$$\begin{aligned}\Psi_\alpha(z) &= c(1-\alpha)S_\alpha(z) + \alpha h E[(S_\alpha(z) - z_2)^+|z_1 = z] \\ &+\alpha p E[(S_\alpha(z) - z_2)^-|z_1 = z] + \alpha c E[z_2|z_1 = z] \\ &+\alpha E\left[\int_0^{(S_\alpha(z) - z_2)^+} \chi_\alpha(\xi, z_2) d\xi | z_1 = z\right]\end{aligned}$$

We have

$$0 \leq \Psi_\alpha(z) \leq m_0 \max(h,c)\left(1 + \frac{1}{1 - \delta_0(m_0)}\right) + (p+c)c_1$$

Consider

$$\Psi(z) = hE[(S(z)-z_2)^+|z_1=z] + pE[(S(z)-z_2)^-|z_1=z]$$
$$+cE[z_2|z_1=z] + E\left[\int_0^{(S(z)-z_2)^+} \chi(\xi, z_2)d\xi|z_1=z\right]$$

and $\Gamma(z)$ to be the solution of

$$\Gamma(z) + \rho = \Psi(z) + E[\Gamma(z_2)|z_1=z],$$

with

$$\rho = \int \Psi(z)\varpi(z)dz.$$

From ergodic theory, see Chapter 3, Theorem 3.3 we can assert that

$$\sup_z |\Gamma(z)| \le \sup_z |\Psi(z)-\rho| \frac{3-\beta}{1-\beta},$$

where $0<\beta<1$, depends only on the Markov chain.

Similarly, if we set

$$\rho_\alpha = \int \Psi_\alpha(z)\varpi(z)dz, \qquad \tilde{\Gamma}_\alpha(z) = \Gamma_\alpha(z) - \frac{\rho_\alpha}{1-\alpha}$$

we can write

$$\tilde{\Gamma}_\alpha(z) + \rho_\alpha = \Psi_\alpha(z) + \alpha E[\tilde{\Gamma}_\alpha(z_2)|z_1=z],$$

we can also assert that

$$\begin{aligned} \sup_z |\tilde{\Gamma}_\alpha(z)| &\le \sup_z |\Psi_\alpha(z)-\rho_\alpha| \frac{3-\beta}{1-\beta} \\ &\le \left[m_0 \max(h,c)\left(1+\frac{1}{1-\delta_0(m_0)}\right) + (p+c)c_1\right]\frac{3-\beta}{1-\beta}. \end{aligned}$$

Moreover

$$\tilde{\Gamma}_\alpha(z) - \tilde{\Gamma}_\alpha(z') = \Psi_\alpha(z) - \Psi_\alpha(z') + \alpha \int \tilde{\Gamma}_\alpha(\zeta)(f(\zeta|z)-f(\zeta|z'))d\zeta.$$

Since, using (13.2.33)

$$|\Psi_\alpha(z)-\Psi_\alpha(z')| \le C(m_0)\delta|z-z'|,$$

and from (12.3.32) we can assert that

$$|\tilde{\Gamma}_\alpha(z) - \tilde{\Gamma}_\alpha(z')| \le C_1(m_0)\delta|z-z'|.$$

Therefore the functions $\tilde{\Gamma}_\alpha(z)$ are uniformly Lipschitz continuous and bounded. It follows that for a subsequence we obtain

$$\sup_{0\le z\le N} \Gamma_\alpha \left|(z) - \frac{\rho_\alpha}{1-\alpha} - \Gamma(z)\right| \to 0, \ \forall M \tag{12.3.52}$$

Therefore

$$\sup_{\substack{x \le M \\ 0 \le z \le N}} \left|u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)\right| \to 0, \forall x, \forall M, N$$

with

$$u(x,z) = hx^+ + px^- - cx + \int_0^x \chi(\xi,z)d\xi + \Gamma(z). \tag{12.3.53}$$

We deduce $u(0,z) = \Gamma(z)$. Moreover, one checks easily that

$$\Gamma(z) + \rho = cS(z) + E[u(S(z) - z_2, z_2)|z_1 = z].$$

Therefore the first relation (13.2.28) is proven.

Consider now the situation with $x \geq S(z)$. Define the function

$$\tilde{u}(x,z) = hx^+ + px^- + E[u(x - z_2, z_2)|z_1 = z].$$

We obtain

$$\begin{aligned}\tilde{u}'(x,z) &= h\mathbb{1}_{x>0} - p\mathbb{1}_{x<0} + E[u'(x - z_2, z_2)|z_1 = z] \\ &= h\mathbb{1}_{x>0} - p\mathbb{1}_{x<0} - c + \chi(x,z), \quad x \geq S(z) \\ &= u'(x,z), \quad x \geq S(z).\end{aligned}$$

Also

$$\begin{aligned}\tilde{u}(S(z),z) &= hS(z) + E[u(S(z) - z_2, z_2)|z_1 = z] \\ &= (h-c)S(z) + \rho + \Gamma(z) \\ &= u(S(z),z) + \rho\end{aligned}$$

From these two relations we get $\tilde{u}(x,z) = u(x,z) + \rho, \forall x \geq S(z)$.

This concludes the second part of (13.2.28).

The proof of (12.3.40) is similar to that of Theorem 12.3.

We have next

$$u_\alpha(x,z) - u_\alpha(x,z') = \Gamma_\alpha(z) - \Gamma_\alpha(z') + \int_0^x (\chi_\alpha(\xi,z) - \chi_\alpha(\xi,z')d\xi$$

Using already proven estimates we obtain

$$|u_\alpha(x,z) - u_\alpha(x,z')| \leq \left[C_1(m_0) + \frac{C(m_0)x^+}{1 - \delta_0(x)}\right] \delta|z - z'|$$

The limit $u(x,z)$ satisfies all the estimates (12.3.37), (12.3.39). The proof of uniqueness is similar to that of Theorem 12.3.

The proof has been completed. □

12.4. NO BACKLOG AND SET UP COST

12.4.1. MODEL. We now study the situation of set up cost, and we begin with the no shortage model. We have to study the Bellman equation

$$u(x,z) = \inf_{v \geq 0}[K\mathbb{1}_{v>0} + l(x,z,v) + \alpha\Phi^v u(x,z)], \tag{12.4.1}$$

where

$$\begin{aligned}\Phi^v\varphi(x,z) &= E[\varphi((x+v-z_2)^+, z_2)|z_1 = z] \\ &= \int_0^{x+v} \varphi(x+v-\zeta,\zeta)f(\zeta|z)d\zeta + \int_{x+v}^\infty \varphi(0,\zeta)f(\zeta|z)d\zeta\end{aligned} \tag{12.4.2}$$

and

(12.4.3) $\quad f(\zeta|z)$ is uniformly continuous in both variables and bounded

$$\int_0^{+\infty} \zeta f(\zeta|z)d\zeta \leq c_0 z + c_1 \tag{12.4.4}$$

$$l(x,z,v) = cv + hx + pE[(x+v-z_2)^-|z_1 = z]. \tag{12.4.5}$$

We look for solutions of (12.4.1) in the interval $[0, w_0(x,z)]$ with

$$w_0(x,z) = l_0(x,z) + \alpha E[w_0(x - z_2, z_2)|z_1 = z]. \tag{12.4.6}$$

In Lemma 12.1, we have proven the estimate

$$w_0(x,z) \leq \frac{hx}{1-\alpha} + \frac{pc_0 z}{1 - c_0\alpha} + \frac{pc_1}{(1-\alpha)(1-c_0\alpha)}, \tag{12.4.7}$$

with the assumption

$$c_0\alpha < 1. \tag{12.4.8}$$

As usual we consider the payoff function

$$J_{x,z}(V) = E\sum_{n=1}^{\infty} \alpha^{n-1}[K\mathbb{1}_{v_n>0} + l(y_n, z_n, v_n)], \tag{12.4.9}$$

with $V = (v_1, \cdots, v_n, \cdots)$ adapted process with positive values and

$$y_{n+1} = (y_n + v_n - z_{n+1})^+,\ y_1 = x. \tag{12.4.10}$$

We define the value function

$$u(x,z) = \inf_V J_{x,z}(V). \tag{12.4.11}$$

We state the

Theorem 12.7. *We assume* (12.4.2)*,* (12.4.3)*,* (12.4.4)*,* (12.4.5)*,* (12.4.8)*. The value function defined in* (12.4.11) *is the unique l.s.c. solution of the Bellman equation* (12.4.1) *in the interval* $[0, w_0]$*. There exists an optimal feedback* $\hat{v}(x,z)$*.*

12.4.2. $s(z), S(z)$ **Policy.** We now prove the following result

Theorem 12.8. *We make the assumptions of Theorem 12.7 and* $p > c$*. Then the function* $u(x,z)$ *is* $K-$*convex and continuous. It tends to* $+\infty$ *as* $x \to +\infty$*. The function*

$$G(x,z) = cx + pE[(x - z_2)^-|z_1 = z] + \alpha E[u((x - z_2)^+, z_2)|z_1 = z],$$

is also $K-$*convex and continuous. It tends to* $+\infty$ *as* $x \to +\infty$*. Considering the numbers* $s(z), S(z)$ *associated to* $G(x,z)$*, the optimal feedback is given by*

$$\hat{v}(x,z) = \begin{cases} S(z) - x, & \text{if } x \leq s(z) \\ 0, & \text{if } x > s(z) \end{cases} \tag{12.4.12}$$

The functions $s(z)$ $S(z)$ *are continuous.*

PROOF. We consider the increasing sequence

$$u_{n+1}(x,z) = \inf_{v\geq 0}\{K\mathbb{1}_{v>0} + cv + hx + pE[(x + v - z_2)^-|z_1 = z] \\ +\alpha E[u_n((x + v - z_2)^+, z_2)|z_1 = z]\}, \tag{12.4.13}$$

with $u_0(x,z) = 0$. Define

$$G_n(x,z) = cx + pE[(x - z_2)^-|z_1 = z] + \alpha E[u_n((x - z_2)^+, z_2)|z_1 = z]. \tag{12.4.14}$$

We can write

$$u_{n+1}(x,z) = (h - c)x + \inf_{\eta \geq x}[K\mathbb{1}_{\eta \geq x} + G_n(\eta, z)]. \tag{12.4.15}$$

We are going to chow, by induction, that both $u_n(x,z), G_n(x,z)$ are $K-$convex in x, continuous and $\to +\infty$ as $x \to +\infty$, for $n \geq 1$. The properties are clear for

$n = 1$. We assume they are verified for n, we prove them for $n+1$. Since $G_n(x,z)$ is $K-$convex in x, continuous and $\to +\infty$ as $x \to +\infty$, we can define $s_n(z), S_n(z)$ with

$$G_n(S_n(z), z) = \inf_{\eta} G_n(\eta, z) \tag{12.4.16}$$

(12.4.17)

$$s_n(z) = \begin{cases} 0, & \text{if } G_n(0,z) \leq K + \inf_\eta G_n(\eta, z) \\ G_n(s_n(z), z) = K + \inf_\eta G_n(\eta, z), & \text{if } G_n(0,z) > K + \inf_\eta G_n(\eta, z) \end{cases}$$

As usual we take the smallest minimum to define $S_n(z)$ in a unique way. Since $G_n(\eta, z)$ is continuous, it is easy to check that $S_n(z)$ is continuous. Also $s_n(z)$ is continuous.

We can write

$$u_{n+1}(x,z) = (h-c)x + G_n(\max(x, s_n(z)), z),$$

which shows immediately that $u_{n+1}(x,z)$ is $K-$convex and continuous. Furthermore $u_{n+1}(x,z) \to +\infty$, as $x \to +\infty$. We then have

$$G_{n+1}(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha(h-c)E[(x-z_2)^+|z_1 = z]$$
$$+\alpha E[G_n(\max(x-z_2, s_n(z_2)), z_2)|z_1 = z]$$

It is the sum of a convex function and a $K-$convex function, hence $K-$convex. It is continuous and $\to +\infty$ as $x \to +\infty$. So the recurrence is proven. If we write (formally)

$$u_n'(x,z) = h - c + \chi_n(x,z),$$

then one has the recurrence

$$\begin{aligned} \chi_{n+1}(x,z) &= 0 \quad \text{if} \quad x < s_n(z) \\ &= \mu(x,z) + \alpha E[\chi_n(x-z_2, z_2)|z_1 = z] \end{aligned}$$

with

$$\mu(x,z) = c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x|z). \tag{12.4.18}$$

The function $\chi_{n+1}(x,z)$ is discontinuous in $s_n(z)$. However one has the bound

$$-\frac{(p-c)}{1-\alpha} \leq \chi_{n+1}(x,z) \leq \frac{c + \alpha(h-c)}{1-\alpha} \quad \text{a.e.}$$

Therefore the limit $u(x,z)$ is $K-$convex and satisfies

$$u'(x,z) = h - c + \chi(x,z),$$

with

$$-\frac{(p-c)}{1-\alpha} \leq \chi(x,z) \leq \frac{c + \alpha(h-c)}{1-\alpha} \quad \text{a.e.}$$

Hence $u(x,z)$ is continuous in x and $\to +\infty$ as $x \to +\infty$. Therefore one defines uniquely $s(z), S(z)$ and $S(z)$ minimizes in x the function

$$G(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha E[u((x-z_2)^+, z_2)|z_1 = z]$$

From this formula and the Lipschitz continuity of u in x, using the assumption (12.4.3) one can see that $G(x,z)$ is continuous in both arguments. Hence $S(z)$ and $s(z)$ are continuous. The proof has been completed. $\square$

12.4.3. ERGODIC THEORY. We now study the behavior of $u(x,z)$ as $\alpha \to +\infty$. We denote it by $u_\alpha(x,z)$ and we write the relations

$$u_\alpha(x,z) = (h-c)x + G_\alpha(\max(x, s_\alpha(z)), z) \tag{12.4.19}$$

$$G_\alpha(x,z) = g_\alpha(x,z) + \alpha E[G_\alpha(\max((x-z_2)^+, s_\alpha(z_2)), z_2)|z_1 = z], \tag{12.4.20}$$

with

$$g_\alpha(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha(h-c)E[(x-z_2)^+|z_1 = z]. \tag{12.4.21}$$

We will use an approach different from that of the base stock case, since we cannot prove uniform Lipschitz properties for the function $s_\alpha(z)$. The present method will use less assumptions. We shall assume $c_0 = 0$ and

$$z_n \quad \text{is ergodic.} \tag{12.4.22}$$

We denote by $\varpi(z)$ the invariant measure. We also assume

$$\begin{aligned} \int |f(\zeta|z) - f(\zeta|z')|d\zeta &\leq \delta|z-z'|, \\ \int \zeta|f(\zeta|z) - f(\zeta|z')|d\zeta &\leq \delta|z-z'| \end{aligned} \tag{12.4.23}$$

$$\sup_z F(x|z) = \delta_0(x) < 1, \forall x \tag{12.4.24}$$

$$\sup_z \bar{F}(x|z) \to 0 \text{ as } x \to +\infty \tag{12.4.25}$$

We begin with the

Lemma 12.3. *If* $s(z) > 0$, *then*

$$S(z) \leq \frac{p + \alpha(h-c)}{c + \alpha(h-c)} E[z_2|z]$$

Proof. If $s(z) > 0$ then we have

$$u(0,z) = K + G(S(z), z).$$

Set

$$\hat{y}_2 = (S(z) - z_2)^+, \quad \hat{v}_2 = \hat{v}(\hat{y}_2, z_2),$$

then

$$G(S(z), z) = cS(z) + pE[(S(z) - z_2)^-|z_1 = z] + \alpha E[u(\hat{y}_2, z_2)|z_1 = z]$$

Also we can write

$$\begin{aligned} u(\hat{y}_2, z_2) &= (h-c)\hat{y}_2 + K\mathbb{1}_{\hat{v}_2 > 0} + G(\hat{y}_2 + \hat{v}_2, z_2) \\ &= h\hat{y}_2 + K\mathbb{1}_{\hat{v}_2 > 0} + c\hat{v}_2 + pE[(\hat{y}_2 + \hat{v}_2 - z_3)^-|z_2] \\ &\quad + \alpha E[u((\hat{y}_2 + \hat{v}_2 - z_3)^+, z_3)|z_2] \end{aligned}$$

Therefore, we can write

$$\begin{aligned} u(0,z) = K &+ cS(z) + pE[(S(z) - z_2)^-|z_1 = z] \\ &+ \alpha E[h\hat{y}_2 + K\mathbb{1}_{\hat{v}_2 > 0} + c\hat{v}_2 + p(\hat{y}_2 + \hat{v}_2 - z_3)^- \\ &+ \alpha u((\hat{y}_2 + \hat{v}_2 - z_3)^+, z_3)|z_1 = z]. \end{aligned} \tag{12.4.26}$$

We next have

$$\begin{aligned} u(0,z) &\leq G(0,z) \\ &= E[pz_2 + \alpha u(0,z_2)|z_1 = z] \end{aligned}$$

Furthermore

$$u(0,z_2) \leq K + G(\hat{y}_2 + \hat{v}_2, z_2)$$

Replacing G and combining the two inequalities we get

$$u(0,z) \leq pE[z_2|z_1 = z] + \alpha K + \alpha E[c(\hat{y}_2 + \hat{v}_2) + p(\hat{y}_2 + \hat{v}_2 - z_3)^-$$
$$+\alpha u((\hat{y}_2 + \hat{v}_2 - z_3)^+, z_3)|z_1 = z]$$

Comparing with (12.4.26) we obtain easily the desired inequality. □

We deduce from the Lemma that, whenever $c_0 = 0$

$$s_\alpha(z) \leq \frac{p + (h-c)^+}{\min(c,h)} c_1. \tag{12.4.27}$$

We now state the

Theorem 12.9. *We assume* (12.4.3), (12.4.4), *with* $c_0 = 0$, (12.4.22), (12.4.23), (12.4.24), (12.4.25). *Then there exists a number* ρ_α *such that, for a subsequence, still denoted* $\alpha \to 1$

$$\sup_{\substack{x \leq M \\ z}} u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)| \to 0, \forall M, \tag{12.4.28}$$

and $\rho_\alpha \to \rho$. *The function* $u(x,z)$ *is Lipschitz continuous,* $K-$*convex and satisfies*

$$|u(x,z)| \leq C_0 + \frac{C_1 x}{1 - \delta_0(x)} \tag{12.4.29}$$

$$|u_x(x,z)| \leq \frac{C}{1-\delta_0(x)}, \quad |u_z(x,z)| \leq C_0 + \frac{C_1 x}{1-\delta_0(x)} \ a.e.. \tag{12.4.30}$$

The pair $u(x,z), \rho$ *is the solution of*

$$u(x,z) + \rho = \inf_{v \geq 0} [K 1\!\!I_{v>0} + l(x,z,v) + \Phi^v u(x,z)]. \tag{12.4.31}$$

PROOF. We set

$$\chi_\alpha(x,z) = u'_\alpha(x,z) - h + c,$$

then we can write

$$\chi_\alpha(x,z) = 1\!\!I_{x > s_\alpha(z)} \mu_\alpha(x,z) + \alpha E[\chi_\alpha(x - z_2, z_2) 1\!\!I_{x - z_2 > 0} | z_1 = z], \tag{12.4.32}$$

with

$$\begin{aligned} \mu_\alpha(x,z) &= g'_\alpha(x,z) \\ &= c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x|z) \end{aligned}$$

The function $\chi_\alpha(x,z)$ is not continuous, but satisfies

$$\sup_{\substack{0 \leq \xi \leq x \\ z}} |\chi_\alpha(\xi,z)| \leq \frac{\max(h,p)}{1-\delta_0(x)}. \tag{12.4.33}$$

Let us next define $\Gamma_\alpha(z) = u_\alpha(0,z)$. We have

$$\begin{aligned}
\Gamma_\alpha(z) &= G_\alpha(s_\alpha(z),z) \\
&= cs_\alpha(z) + pE[(s_\alpha(z)-z_2)^-|z_1=z] + \alpha E[u_\alpha((s_\alpha(z)-z_2)^+,z_2)|z_1=z] \\
&= g_\alpha(s_\alpha(z),z)+\alpha E\left[\int_0^{(s_\alpha(z)-z_2)^+}\chi_\alpha(\xi,z_2)d\xi|z_1=z\right]+\alpha E[\Gamma_\alpha(z_2)|z_1=z]
\end{aligned}$$

Therefore we can write

$$\Gamma_\alpha(z) = \Psi_\alpha(z) + \alpha E[\Gamma_\alpha(z_2)|z_1=z], \tag{12.4.34}$$

with

$$\Psi_\alpha(z) = g_\alpha(s_\alpha(z),z) + \alpha E\left[\int_0^{(s_\alpha(z)-z_2)^+}\chi_\alpha(\xi,z_2)d\xi|z_1=z\right].$$

Thanks to Lemma 12.3, we have the estimate (12.4.27), so $s_\alpha(z) \le m_0$. Using the estimate (12.4.33), we see that $|\Psi_\alpha(z)| \le C$. Let us then define

$$\rho_\alpha = \int \Psi_\alpha(z)\varpi(z)dz,$$

and $\tilde{\Gamma}_\alpha(z) = \Gamma_\alpha(z) - \dfrac{\rho_\alpha}{1-\alpha}$. From ergodic theory, we can assert that

$$\sup_z |\tilde{\Gamma}_\alpha(z)| \le \sup_z |\Psi_\alpha(z) - \rho_\alpha|\frac{3-\beta}{1-\beta} \le C,$$

and we can state the estimate

$$\left|u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha}\right| \le C + \frac{Cx}{1-\delta_0(x)}.$$

Define $\tilde{u}_\alpha(x,z) = u_\alpha(x,z) - \dfrac{\rho_\alpha}{1-\alpha}$. We write equation (12.4.1) as

$$\tilde{u}_\alpha(x,z) + \rho_\alpha = \inf_{v\ge 0}[K\mathbb{1}_{v>0} + l(x,z,v) + \alpha\Phi^v\tilde{u}_\alpha(x,z)]. \tag{12.4.35}$$

From Lemma 12.3 we can assert that the optimal feedback satisfies $x + \hat{v}_\alpha(x,z) \le \max(x,m_0)$. Therefore we can replace (12.4.35) by

$$\begin{aligned}
\tilde{u}_\alpha(x,z) + \rho_\alpha &= \inf_{x+v\le\max(x,m_0)}[K\mathbb{1}_{v>0} + l(x,z,v) + \alpha\Phi^v\tilde{u}_\alpha(x,z)] \\
&= \inf_{0\le v\le\max(x,m_0)-x} L_\alpha(x,z,v).
\end{aligned}$$

Next

$$L_\alpha(x,z,v) - L_\alpha(x,z',v) = \int[p(x+v-\zeta)^- + \alpha\tilde{u}_\alpha((x+v-\zeta)^+,\zeta)](f(\zeta|z) - f(\zeta|z'))d\zeta$$

For $v \le \max(x,m_0) - x$ we can write

$$\begin{aligned}
|\tilde{u}_\alpha((x+v-\zeta)^+,\zeta)| &\le C + \frac{C\max(x,m_0)}{1-\delta_0(\max(x,m_0))} \\
&\le C_1' + \frac{C_2x}{1-\delta_0(x)}
\end{aligned}$$

Using the assumption (12.4.23) we get easily

$$|\tilde{u}_\alpha(x,z) - \tilde{u}_\alpha(x,z')| \le \left(C_1 + \frac{C_2x}{1-\delta_0(x)}\right)\delta|z-z'|.$$

From the estimates obtained we can assert that $\tilde{u}_\alpha(x,z)$ has a converging subsequence (still denoted α), in the sense

$$\sup_{\substack{x \le M \\ z \le N}} |\tilde{u}_\alpha(x,z) - u(x,z)| \to 0, \text{as } \alpha \to 0. \tag{12.4.36}$$

Since ρ_α is bounded, we can always assume that $\rho_\alpha \to \rho$. Denote

$$L(x,z,v) = K\mathbb{1}_{v>0} + l(x,z,v) + \Phi^v u(x,z)$$

then

$$L_\alpha(x,z,v) - L(x,z,v) = \alpha\Phi^v(\tilde{u}_\alpha - u)(x,z) - (1-\alpha)\Phi^v u(x,z)$$

For $v \le \max(x, m_0) - x$, we have assuming $M > m_0$

$$\sup_{\substack{x \le M \\ z}} |\Phi^v u(x,z)| \le C + \frac{CM}{1-\delta_0(M)}$$

We also have

$$\begin{aligned} \sup_{\substack{x \le M \\ z}} |\Phi^v(\tilde{u}_\alpha - u)(x,z)| \quad &\le \quad \sup_{\substack{x \le M \\ z}} |\tilde{u}_\alpha - u|(x,z) \sup_z \bar{F}(N,z) \\ &+ \sup_{\substack{x \le M \\ z \le N}} |\tilde{u}_\alpha - u|(x,z) \end{aligned}$$

Using the assumption (12.4.25) and (12.4.36), letting first $\alpha \to 1$, then $N \to \infty$, we deduce

$$\sup_{\substack{x \le M \\ v \le \max(x,m_0) - x \\ z}} |L_\alpha(x,z,v) - L(x,z,v)| \to 0, \text{ as } \alpha \to 1$$

Therefore we deduce easily (12.4.28) and also that the pair $u(x,z), \rho$ satisfy (12.4.31). The estimates (12.4.29), (12.4.31) follow immediately from the corresponding ones on $\tilde{u}_\alpha(x,z)$. The $K-$convexity of u follows from the $K-$convexity of $\tilde{u}_\alpha$. The proof has been completed. □

12.5. BACKLOG AND SET UP COST

12.5.1. MODEL. We finally consider the situation of set up cost, with backlog. The Bellman equation is

$$u(x,z) = \inf_{v \ge 0}[K\mathbb{1}_{v>0} + l(x,v) + \alpha\Phi^v u(x,z)], \tag{12.5.1}$$

where

$$\begin{aligned} \Phi^v\varphi(x,z) \quad &= \quad E[\varphi(x+v-z_2, z_2)|z_1 = z] \\ &= \quad \int_0^\infty \varphi(x+v-\zeta,\zeta) f(\zeta|z) d\zeta, \end{aligned} \tag{12.5.2}$$

and

(12.5.3) $f(\zeta|z)$ is uniformly continuous in both variables and bounded

$$\int_0^{+\infty} \zeta f(\zeta|z)d\zeta \leq c_0 z + c_1, \tag{12.5.4}$$

$$l(x,v) = cv + hx^+ + px^-. \tag{12.5.5}$$

We look for solutions of (12.5.1) in the interval $[0, w_0(x,z)]$ with

$$w_0(x,z) = l_0(x) + \alpha E[w_0(x - z_2, z_2)|z_1 = z], \tag{12.5.6}$$

with $l_0(x) = l(x,0)$.

In Lemma 12.2, we have proven the estimate

$$w_0(x,z) \leq \frac{hx^+ + px^-}{1-\alpha} + \frac{pc_0 z\alpha}{(1-c_0\alpha)(1-\alpha)} + \frac{pc_1}{(1-c_0)^2}\left(\frac{\alpha - c_0}{(1-\alpha)^2} + \frac{c_0}{1-\alpha c_0}\right), \tag{12.5.7}$$

with the assumption

$$c_0\alpha < 1. \tag{12.5.8}$$

As usual we consider the payoff function

$$J_{x,z}(V) = E\sum_{n=1}^{\infty} \alpha^{n-1}[K1\!\!1_{v_n>0} + l(y_n, v_n)], \tag{12.5.9}$$

with $V = (v_1, \cdots, v_n, \cdots)$ adapted process with positive values and

$$y_{n+1} = y_n + v_n - z_{n+1},\ y_1 = x. \tag{12.5.10}$$

We define the value function

$$u(x,z) = \inf_V J_{x,z}(V). \tag{12.5.11}$$

We state the

Theorem 12.10. *We assume* (12.5.2), (12.5.3), (12.5.4), (12.5.5), (12.5.8). *The value function defined in* (12.5.11) *is the unique l.s.c. solution of the Bellman equation* (12.5.1) *in the interval* $[0, w_0]$. *There exists an optimal feedback* $\hat{v}(x,z)$.

12.5.2. $s(z), S(z)$ **Policy.** We now prove the following result

Theorem 12.11. *We make the assumptions of Theorem 12.10 and* $\alpha p > c$. *Then the function* $u(x,z)$ *is* $K-$*convex and continuous. It tends to* $+\infty$ *as* $|x| \to +\infty$. *Considering the numbers* $s(z), S(z)$ *associated to* $u(x,z)$, *the optimal feedback is given by*

$$\hat{v}(x,z) = \begin{cases} S(z) - x, & \text{if } x \leq s(z) \\ 0, & \text{if } x > s(z) \end{cases} \tag{12.5.12}$$

The functions $s(z)$ $S(z)$ *are continuous,* $S(z) > 0$.

PROOF. We consider the increasing sequence

$$u_{n+1}(x,z) = hx^+ + px^- + \inf_{v\geq 0}\{K1\!\!1_{v>0} + cv + \alpha E[u_n(x+v-z_2, z_2)|z_1 = z]\}, \tag{12.5.13}$$

with $u_0(x,z) = 0$. Define

$$G_n(x,z) = cx + \alpha E[u_n(x - z_2, z_2)|z_1 = z]. \tag{12.5.14}$$

We can write

$$u_{n+1}(x,z) = hx^+ + px^- - cx + \inf_{\eta \geq x}[K1\!\!1_{\eta > x} + G_n(\eta, z)]. \tag{12.5.15}$$

We are going to chow, by induction, that both $u_n(x,z), G_n(x,z)$ are $K-$convex in x, continuous and $\to +\infty$ as $|x| \to +\infty$, for $n \geq 2$. Also

$$u_n'(x,z) \leq -p-c,\ G_n'(x,z) \leq c(1-\alpha)-\alpha p,\ \text{if } x<0. \tag{12.5.16}$$

These relations will be true for $n \geq 2$. We have $G_0'(x,z) = c$ and $u_1'(x,z) = -p$, if $x < 0$. Then

$$G_1'(x,z) = c - \alpha p < 0,\ \text{if } x < 0$$

As soon as $G_n'(x,z) \leq 0$, for $x<0$, then $u_{n+1}'(x,z)$ satisfies (12.5.16) and $G_{n+1}'(x,z)$ satisfies also (12.5.16).

Since $G_n(x,z)$ is $K-$convex in x, continuous and $\to +\infty$ as $|x| \to +\infty$, we can define $s_n(z), S_n(z)$ with

$$G_n(S_n(z),z) = \inf_\eta G_n(\eta,z), \tag{12.5.17}$$

and $s_n(z)$ is defined by

$$G_n(s_n(z),z) = K + \inf_\eta G_n(\eta,z). \tag{12.5.18}$$

As usual we take the smallest minimum to define $S_n(z)$ in a unique way. Since $G_n(\eta,z)$ is continuous, it is easy to check that $S_n(z)$ is continuous. Also $s_n(z)$ is continuous.

We can write

$$u_{n+1}(x,z) = hx^+ + px^- - cx + G_n(\max(x,s_n(z)),z)$$

which shows immediately that $u_{n+1}(x,z)$ and then $G_{n+1}(x,z)$ are $K-$convex and continuous. Furthermore $u_{n+1}(x,z), G_{n+1}(x,z) \to +\infty$, as $x \to +\infty$.

Since for $x<0$

$$u_{n+1}'(x,z) = -p - c + G_n'(x,z)\mathbb{1}_{x>s_n(z)},$$

we see that (12.5.16) is satisfied for u_{n+1}, G_{n+1}. Therefore $u_{n+1}(x,z), G_{n+1}(x,z) \to +\infty$ as $x \to -\infty$.

So the recurrence is proven. Note also that $S_n(z) > 0$.

If we write (formally)

$$u_n'(x,z) = h\mathbb{1}_{x>0} - p\mathbb{1}_{x<0} - c + \chi_n(x,z),$$

then one has the recurrence

$$\begin{aligned}\chi_{n+1}(x,z) &= 0 \quad \text{if} \quad x < s_n(z) \\ &= \mu(x,z) + \alpha E[\chi_n(x-z_2,z_2)|z_1 = z]\end{aligned}$$

with

$$\mu(x,z) = c(1-\alpha) + \alpha h - \alpha(p+h)\bar{F}(x|z). \tag{12.5.19}$$

The function $\chi_{n+1}(x,z)$ is discontinuous in $s_n(z)$. However one has the bound

$$\frac{c(1-\alpha)-\alpha p}{1-\alpha} \leq \chi_{n+1}(x,z) \leq \frac{c(1-\alpha)+\alpha h}{1-\alpha} \quad \text{a.e.}$$

Note also $\chi_n(x,z) \leq 0$, if $x<0$ and

$$\chi_{n+1}(x,z) \leq c(1-\alpha) - \alpha p,\ \text{if } s_n(z) < x < 0$$

Therefore the limit $u(x,z)$ is $K-$convex and satisfies

$$u'(x,z) = h\mathbb{1}_{x>0} - p\mathbb{1}_{x<0} - c + \chi(x,z),$$

with

$$\frac{c(1-\alpha)-\alpha p}{1-\alpha} \leq \chi(x,z) \leq \frac{c(1-\alpha)+\alpha h}{1-\alpha} \quad \text{a.e.}$$

Hence $u(x,z)$ is continuous in x and $\to +\infty$ as $x \to +\infty$. For $x<0$, one also has

$$u'(x,z) \leq -c-p$$

therefore $u(x,z) \to +\infty$ as $x \to +\infty$. Also

$$\chi(x,z) \leq 0 \text{ if } x<0, \;\; \chi(x,z) \leq c(1-\alpha)-\alpha p, \; s(z)<x<0$$

Therefore there one defines uniquely $s(z), S(z)$ and $S(z)$ minimizes in x the function

$$G(x,z) = cx + \alpha E[u(x-z_2, z_2)|z_1 = z]$$

From this formula and the Lipschitz continuity of u in x, using the assumption (12.5.3) one can see that $G(x,z)$ is continuous in both arguments.

Also $G(x,z) \to +\infty$ as $x \to -\infty$. Hence $S(z)$ and $s(z)$ are well defined, and are continuous. Moreover $S(z)>0$. The proof has been completed. □

12.5.3. ERGODIC THEORY. We now study the behavior of $u(x,z)$ as $\alpha \to +\infty$. We denote it by $u_\alpha(x,z)$ and we write the relations

(12.5.20) $$u_\alpha(x,z) = hx^+ + px^- - cx + G_\alpha(\max(x, s_\alpha(z)), z)$$

(12.5.21) $$G_\alpha(x,z) = cx + \alpha E[u_\alpha(x-z_2, z_2)|z_1 = z].$$

We shall assume $c_0 = 0$ and

(12.5.22) $$z_n \quad \text{is ergodic.}$$

We denote by $\varpi(z)$ the invariant measure. We also assume

(12.5.23) $$\int |f(\zeta|z) - f(\zeta|z')|d\zeta \quad \leq \quad \delta|z-z'|,$$

(12.5.24) $$\int \zeta|f(\zeta|z) - f(\zeta|z')|d\zeta \leq \delta|z-z'|,$$

(12.5.25) $$\sup_z F(x|z) \quad = \quad \delta_0(x) < 1, \forall x$$

(12.5.26) $$\sup_z \bar{F}(x|z) \quad \to \quad 0 \text{ as } x \to +\infty$$

We begin with the

Lemma 12.4. *We have the estimate*

(12.5.27) $$s_\alpha^-(z) \leq \frac{K}{\alpha p - c(1-\alpha)} = m_\alpha$$

Also

(12.5.28) $$0 \leq S_\alpha(z) \leq \frac{\alpha(p+h)E[z_2|z_1=z] + \alpha p m_\alpha}{c + \alpha(h-c)}$$

PROOF. We recall that

$$0 = K + \inf_{\eta > s(z)} \int_{s(z)}^{\eta} \chi(\xi, z) d\xi.$$

Therefore if $s(z) < 0$ we can write

$$0 \leq K + \int_{s(z)}^{0} \chi(\xi, z) d\xi$$

However for $s(z) < \xi < 0$, we have $\chi(\xi, z) \leq c(1-\alpha) - \alpha p$, which implies (12.5.27) immediately.

Since $s_\alpha(z) > -m_\alpha$, we have successively

$$u(-m_\alpha, z) = (p+c)m_\alpha + K + G(S(z), z).$$

Set

$$\hat{y}_2 = S(z) - z_2, \quad \hat{v}_2 = \hat{v}(\hat{y}_2, z_2),$$

then

$$G(S(z), z) = cS(z) + \alpha E[u(\hat{y}_2, z_2)|z_1 = z].$$

Also we can write

$$u(\hat{y}_2, z_2) \quad = \quad h\hat{y}_2^+ + p\hat{y}_2^- + K 1\!\!1_{\hat{v}_2 > 0} + c\hat{v}_2 + \alpha E[u(\hat{y}_2 + \hat{v}_2 - z_3, z_3)|z_2].$$

Therefore, we can write

$$\begin{aligned} u(-m_\alpha, z) = (p+c)m_\alpha + K + cS(z) + \alpha E[h\hat{y}_2^+ + p\hat{y}_2^- \\ + K 1\!\!1_{\hat{v}_2 > 0} + c\hat{v}_2 + \alpha u(\hat{y}_2 + \hat{v}_2 - z_3, z_3)|z_1 = z]. \end{aligned} \tag{12.5.29}$$

We next have

$$u(-m_\alpha, z) \quad \leq \quad pm_\alpha + \alpha E[u(-m_\alpha - z_2, z_2)|z_1 = z].$$

Furthermore

$$u(-m_\alpha - z_2, z_2) \leq K + p(z_2 + m_\alpha) + c(\hat{y}_2 + \hat{v}_2 + z_2 + m_\alpha) + \alpha E[u(\hat{y}_2 + \hat{v}_2 - z_3, z_3)|z_2].$$

Combining the two inequalities we get

$$\begin{aligned} u(-m_\alpha, z) \leq pm_\alpha + \alpha K + \alpha(p+c)m_\alpha \\ + \alpha E[(p+c)z_2 + c(\hat{y}_2 + \hat{v}_2) + \alpha u(\hat{y}_2 + \hat{v}_2 - z_3, z_3)|z_1 = z]. \end{aligned}$$

Comparing with (12.5.29) we obtain inequality

$$\begin{aligned} cS(z) + \alpha(h-c)E[(S(z) - z_2)^+|z_1 \quad &= \quad z] + \alpha(p+c)E[(S(z) - z_2)^-|z_1 = z] \\ &\leq \quad \alpha p m_\alpha + \alpha(p+c)E[z_2|z_1 = z], \end{aligned}$$

from which we deduce easily (12.5.28), which completes the proof. □

We deduce from the Lemma that if $\alpha p > c + \alpha(h-c)$ one has also

$$|s_\alpha(z)| \leq \frac{\alpha(p+h)E[z_2|z_1 = z] + \alpha p m_\alpha}{c + \alpha(h-c)}. \tag{12.5.30}$$

Since $\alpha \to 1$, we may assume $\alpha > \dfrac{1}{2}$ and thus if $c_0 = 0$

$$|s_\alpha(z)|, \; S_\alpha(z) \quad \leq \quad m_0 = \frac{(p+h)c_1 + \dfrac{2Kp}{p-c}}{\min(c,h)}$$

We now state the

Theorem 12.12. *We assume* (12.5.3), (12.5.4), *with* $c_0 = 0$, (12.5.22), (12.5.23), (12.5.25), (12.5.26). *Then there exists a number* ρ_α *such that, for a subsequence, still denoted* $\alpha \to 1$

$$\sup_{\substack{|x| \leq M \\ z}} \left| u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z) \right| \to 0, \forall M, \tag{12.5.31}$$

and $\rho_\alpha \to \rho$. *The function* $u(x,z)$ *is Lipschitz continuous,* $K-$*convex and satisfies*

$$|u(x,z)| \leq C + \frac{C(x+m_0)^+}{1-\delta_0((x+m_0)^+)} + C(x+m_0)^- \tag{12.5.32}$$

$$\begin{aligned} |u_x(x,z)| &\leq \frac{C}{1-\delta_0((x+m_0)^+)}, \quad |u_z(x,z)| \\ &\leq C + \frac{C(\max(x^+,m_0)+m_0)}{1-\delta_0(\max(x^+,m_0)+m_0)} \ a.e.. \end{aligned} \tag{12.5.33}$$

The pair $u(x,z), \rho$ *is the solution of*

$$u(x,z) + \rho = \inf_{v \geq 0} [K 1\!\!I_{v>0} + l(x,v) + \Phi^v u(x,z)]. \tag{12.5.34}$$

Proof. We set

$$\chi_\alpha(x,z) = u'_\alpha(x,z) - h1\!\!I_{x>0} + p1\!\!I_{x<0} + c,$$

then we can write

$$\chi_\alpha(x,z) = 1\!\!I_{x>s_\alpha(z)}\{\mu_\alpha(x,z) + \alpha E[\chi_\alpha(x-z_2,z_2)|z_1 = z], \tag{12.5.35}$$

with

$$\mu_\alpha(x,z) \quad = \quad c + \alpha(h-c) - \alpha(h+p)\bar{F}(x|z)$$

The function $\chi_\alpha(x,z)$ is not continuous, but satisfies (since $s_\alpha(z) \geq -m_0$)

$$\sup_{\substack{\xi \leq x \\ z}} |\chi_\alpha(\xi,z)| \leq \frac{\max(h,p)}{1-\delta_0(x^+ + m_0)}. \tag{12.5.36}$$

Let us next define $\Gamma_\alpha(z) = u_\alpha(-m_0,z)$. We have

$$\begin{aligned} \Gamma_\alpha(z) &= G_\alpha(s_\alpha(z),z) + (p+c)m_0 \\ &= cs_\alpha(z) + \alpha E[u_\alpha(s_\alpha(z)-z_2,z_2)|z_1 = z] + (p+c)m_0 \\ &= g_\alpha(s_\alpha(z),z) + \alpha E[\Gamma_\alpha(z_2)|z_1 = z] \\ &\quad +\alpha E\left[\int_{-m_0}^{s_\alpha(z)-z_2} \chi_\alpha(\xi,z_2)d\xi|z_1 = z\right] \end{aligned}$$

with

$$\begin{aligned} g_\alpha(x,z) &= c(1-\alpha)x + \alpha E[h(x-z_2)^+ + p(x-z_2)^-] \\ &\quad +cE[z_2|z_1 = z] + (p+c)m_0(1-\alpha) \end{aligned} \tag{12.5.37}$$

Therefore we can write

$$\Gamma_\alpha(z) = \Psi_\alpha(z) + \alpha E[\Gamma_\alpha(z_2)|z_1 = z], \tag{12.5.38}$$

with

$$\Psi_\alpha(z) = g_\alpha(s_\alpha(z), z) + \alpha E\left[\int_{-m_0}^{s_\alpha(z)-z_2} \chi_\alpha(\xi, z_2)d\xi | z_1 = z\right].$$

We note that $|s_\alpha(z)| \leq m_0$. Using the estimate (12.5.36), we see that $|\Psi_\alpha(z)| \leq C$. Let us then define

$$\rho_\alpha = \int \Psi_\alpha(z)\varpi(z)dz,$$

and $\tilde{\Gamma}_\alpha(z) = \Gamma_\alpha(z) - \frac{\rho_\alpha}{1-\alpha}$. From ergodic theory, we can assert that

$$\sup_z |\tilde{\Gamma}_\alpha(z)| \leq \sup_z |\Psi_\alpha(z) - \rho_\alpha| \frac{3-\beta}{1-\beta} \leq C,$$

and we can state the estimate

$$\left| u_\alpha(x, z) - \frac{\rho_\alpha}{1-\alpha} \right| \leq C + \frac{C(x+m_0)^+}{1-\delta_0((x+m_0)^+)} + C(x+m_0)^-.$$

Define $\tilde{u}_\alpha(x, z) = u_\alpha(x, z) - \frac{\rho_\alpha}{1-\alpha}$. We write equation (12.5.1) as

$$\tilde{u}_\alpha(x, z) + \rho_\alpha = \inf_{v \geq 0}[K\mathbb{1}_{v>0} + l(x, v) + \alpha\Phi^v \tilde{u}_\alpha(x, z)]. \tag{12.5.39}$$

From Lemma 12.4 we can assert that the optimal feedback satisfies

$$\max(-m_0, x) \leq x + \hat{v}(x, z) \leq \max(x^+, m_0).$$

Therefore we can replace (12.5.39) by

$$\begin{aligned} \tilde{u}_\alpha(x, z) + \rho_\alpha &= \inf_{\max(-m_0, x) \leq x+v \leq \max(x^+, m_0)} [K\mathbb{1}_{v>0} + l(x, v) + \alpha\Phi^v \tilde{u}_\alpha(x, z)] \\ &= \inf_{\max(-m_0, x) \leq x+v \leq \max(x^+, m_0)} L_\alpha(x, z, v). \end{aligned}$$

Next

$$L_\alpha(x, z, v) - L_\alpha(x, z', v) = \alpha \int \tilde{u}_\alpha(x+v-\zeta, \zeta)(f(\zeta|z) - f(\zeta|z'))d\zeta$$

For $\max(-m_0, x) \leq x + v \leq \max(x^+, m_0)$ we can write

$$|\tilde{u}_\alpha(x+v-\zeta, \zeta)| \leq C + \frac{C(\max(x^+, m_0) + m_0)}{1-\delta_0(\max(x^+, m_0) + m_0)} + C\zeta$$

Using the assumptions (12.5.23) and (12.5.24) we get easily

$$|\tilde{u}_\alpha(x, z) - \tilde{u}_\alpha(x, z')| \leq \left(C + \frac{C(\max(x^+, m_0) + m_0)}{1-\delta_0(\max(x^+, m_0) + m_0)}\right)\delta|z - z'|.$$

From the estimates obtained we can assert that $\tilde{u}_\alpha(x, z)$ has a converging subsequence (still denoted α), in the sense

$$\sup_{\substack{|x| \leq M \\ z \leq N}} |\tilde{u}_\alpha(x, z) - u(x, z)| \to 0, \text{as } \alpha \to 0. \tag{12.5.40}$$

Since ρ_α is bounded, we can always assume that $\rho_\alpha \to \rho$. Denote

$$L(x, z, v) = K\mathbb{1}_{v>0} + l(x, v) + \Phi^v u(x, z)$$

then

$$L_\alpha(x, z, v) - L(x, z, v) = \alpha\Phi^v(\tilde{u}_\alpha - u)(x, z) - (1-\alpha)\Phi^v u(x, z)$$

For $\max(-m_0, x) \le x + v \le \max(x^+, m_0)$, we have assuming $M > m_0$

$$\sup_{\substack{|x| \le M \\ z}} |\Phi^v u(x,z)| \le C + \frac{CM}{1 - \delta_0(2M)}$$

We also have

$$\begin{aligned} \sup_{\substack{|x| \le M \\ z}} |\Phi^v(\tilde{u}_\alpha - u)(x,z)| \quad &\le \quad \sup_{\substack{|x| \le M \\ z}} |\tilde{u}_\alpha - u|(x,z) \sup_z \bar{F}(N,z) \\ &+ \sup_{\substack{|x| \le M \\ z \le N}} |\tilde{u}_\alpha - u|(x,z) \end{aligned}$$

Using the assumption (12.5.26) and (12.5.40), letting first $\alpha \to 1$, then $N \to \infty$, we deduce

$$\sup_{\substack{|x| \le M \\ \max(-m_0,x) \le v + x \le \max(x^+, m_0) \\ z}} |L_\alpha(x,z,v) - L(x,z,v)| \to 0, \text{ as } \alpha \to 1.$$

Therefore we deduce easily (12.5.31) and also that the pair $u(x,z), \rho$ satisfies (12.5.34). The estimates (12.5.32), (12.5.33) follow immediately from the corresponding ones on $\tilde{u}_\alpha(x,z)$. The $K-$convexity of u follows from the $K-$convexity of $\tilde{u}_\alpha$. The proof has been completed. □

12.6. LEARNING PROCESS

12.6.1. MODEL. We follow in this section [**33**]. We have already considered an example of Markov demand in section 12.2, Example 12.1, in which the Markov dependence was obtained trough a parameter of the probability law of the demand. The probability law of the demand z_{n+1} depends on a parameter, namely $f(x, \lambda)$, but λ is not fixed. It is a function of z_n, $\lambda(z_n)$. We show here how the common learning procedure, which is quite natural when a parameter is not really known, but is updated in function of the available information leads to a similar situation. We consider only the inventory model with backlog, to fix the ideas. So we have

$$y_{n+1} = y_n + v_n - z_{n+1}, \quad y_1 = x. \tag{12.6.1}$$

The probability density of z_{n+1} is $f(z, \lambda_n)$ where λ_n is the value given to the parameter λ, at time n. Instead of writing $\lambda_n = \lambda(z_n)$, we express an evolution

$$\lambda_{n+1} = \rho(\lambda_n, z_{n+1}), \quad \lambda_1 = \lambda. \tag{12.6.2}$$

The pair y_n, λ_n is a controlled Markov chain, whose operator is given by

$$\begin{aligned} \Phi^v \varphi(x,\lambda) &= E[\varphi(x + v - z_2, \rho(\lambda, z_2))] \\ &= \int \varphi(x + v - \zeta, \rho(\lambda,\zeta)) f(\zeta,\lambda) d\zeta \end{aligned} \tag{12.6.3}$$

The observation process is the demand. So we define the filtration

$$\begin{aligned} \mathcal{F}^n &= \sigma(z_2, \cdots, z_n), \; n \ge 2 \\ \mathcal{F}^1 &= (\Omega, \emptyset) \end{aligned}$$

and the control process v_n is adapted to the filtration $\mathcal{F}^n$. We define the cost function for each period

$$l(x,v) = hx^+ + px^- + cv. \tag{12.6.4}$$

Calling $V = \{v_1, \cdots, v_n, \cdots\}$a control policy, we define the cost functional

$$J_{x,\lambda}(V) = E\left[\sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n) | y_1 = x, \lambda_1 = \lambda\right]. \tag{12.6.5}$$

We consider the value function

$$u(x,\lambda) = \inf_V J_{x,\lambda}(V), \tag{12.6.6}$$

and the Bellman equation

$$u(x,\lambda) = hx^+ + px^- - cx + \inf_{\eta \geq x}[c\eta + \alpha\Phi^v u(\eta,\lambda)]. \tag{12.6.7}$$

12.6.2. SCALABILITY. Bellman equation (12.6.7) is, as expected a 2 dimensional problem. We just describe here a case when the problem can be reduced to one dimension, by an interesting property, the *scalability*. We say that there is scalability with the function $q(\lambda)$ if one has the relations

$$f(x,\lambda) = \frac{f\left(\dfrac{x}{q(\lambda)}\right)}{q(\lambda)}, \qquad q(\rho(\lambda,x)) = q(\lambda)U\left(\frac{x}{q(\lambda)}\right). \tag{12.6.8}$$

Let us give examples. The first one is the function

$$f(x,\lambda) = \frac{ka\lambda^a x^{k-1}}{(\lambda + x^k)^{a+1}}. \tag{12.6.9}$$

This probability density is called Weibull-Gamma. The reason is that it is obtained by combining a Weibull distribution with a Gamma distribution. Suppose we consider a Weibull distribution

$$\psi(x,y) = kyx^{k-1} \exp -yx^k,$$

depending on a parameter y. However, the parameter y is not known. One considers that it is random with a density

$$g(y,\lambda) = \frac{\lambda^a y^{a-1} \exp -y\lambda}{\Gamma(a)},$$

where we recall that for $a > 0$

$$\Gamma(a) = \int_0^{+\infty} x^{a-1} \exp -x \, dx.$$

We see immediately that

$$f(x,\lambda) = \int_0^{+\infty} \psi(x,y) g(y,\lambda) dy.$$

The function $f(x,\lambda)$ defined by (12.6.9) is scalable if $\rho(\lambda,x) = \lambda + x^k$. The functions $q(\lambda), f(x), U(x)$ are defined by

$$q(\lambda) = \lambda^{\frac{1}{k}}, \quad f(x) = \frac{kax^{k-1}}{(1+x^k)^{a+1}}, \quad U(x) = (1+x^k)^{\frac{1}{k}}. \tag{12.6.10}$$

A second example is the Gamma-Gamma model in which

$$f(x,\lambda) = \frac{\lambda^a x^{k-1}\Gamma(k+a)}{(x+\lambda)^{k+a}\Gamma(a)\Gamma(k)}. \tag{12.6.11}$$

This probability density can be obtained by combining a Gamma probability density

$$\psi(x,y) = \frac{y^k x^{k-1}\exp -y}{\Gamma(k)},$$

with another Gamma probability density

$$g(y,\lambda) = \frac{\lambda^a y^{a-1}\exp -y\lambda}{\Gamma(a)}.$$

The probability density (12.6.11) is scalable if $\rho(\lambda,x) = \lambda + x$. The functions $q(\lambda), f(x), U(x)$ are defined by

$$q(\lambda) = \lambda, \quad f(x) = \frac{x^{k-1}\Gamma(k+a)}{(1+x)^{k+a}\Gamma(a)\Gamma(k)}, \quad U(x) = 1 + x. \tag{12.6.12}$$

Let us now show how the scalability property can be used to reduce the dimension of Bellman equation to 1.

Equation (12.6.7) is written as

$$u(x,\lambda) = hx^+ + px^- - cx + \inf_{\eta\geq x}\left[c\eta + \alpha\int u(\eta-\zeta,\rho(\lambda,\zeta))f(\zeta,\lambda)d\zeta\right]. \tag{12.6.13}$$

Replacing $f(\zeta,\lambda)$ we get

$$u(x,\lambda) = hx^+ + px^- - cx + \inf_{\eta\geq x}\left[c\eta + \alpha\int u(\eta-\zeta,\rho(\lambda,\zeta))\frac{f\left(\frac{\zeta}{q(\lambda)}\right)}{q(\lambda)}d\zeta\right].$$

We can find a solution of the form

$$u(x,\lambda) = v\left(\frac{x}{q(\lambda)}\right)q(\lambda),$$

with $v(x)$ solution of

$$v(x) = hx^+ + px^- - cx + \inf_{\eta\geq x}\left[c\eta + \alpha\int v\left(\frac{\eta-\zeta}{U(\zeta)}\right)U(\zeta)f(\zeta)d\zeta\right], \tag{12.6.14}$$

which is a one-dimensional problem.

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CHAPTER 13

LEAD TIMES AND DELAYS

13.1. INTRODUCTION

In all models, so far we have considered that an order v_n put at time n is delivered at time $n+1$. Another equivalent way is to consider that n represents the initial time of the n^{th} period and the the delivery is achieved at the end of the period, which is also the beginning of the $(n+1)^{\text{th}}$ period. The delivery time, called the *lead time* is 1. In this chapter, we are going to consider more complex situations, in which the lead time is larger than 1. Many situations can occur, the simplest being that the lead time is a fixed deterministic number L. It can also be a random variable and even a stochastic process. During the lead time, there can be limitations in the orders, for example no order is possible during a lead time, or orders can be put without any limitation.

Concerning delays, we shall consider delays in the information reception. In the preceding chapters, the delay was 0. Indeed, the demand is observed when it is realized; the demand in the n^{th} period is realized at the end of the n^{th} period, which also time $n+1$, when the order v_{n+1} is decided.

In these problems the dynamics is not indifferent. The backlog case, in which the evolution of the inventory is linear, can lead to important simplifications, in which an interesting concept the *inventory position* plays a major role.

13.2. MODELS WITH INVENTORY POSITION

13.2.1. DETERMINISTIC LEAD TIME. We assume here that we have a lead time $L \geq 1$. Of course, $L = 1$ is the standard case. The demand is a sequence of i.i.d random variables, $D_1, \cdots, D_n, \cdots$ We define the filtration

$$\mathcal{F}^n = \sigma(D_1, \cdots, D_n), \quad \mathcal{F}^0 = (\Omega, \emptyset).$$

The control $V = (v_1, \cdots, v_n, \cdots)$ is adapted to the the filtration $\mathcal{F}^{n-1}$. If we denote by $(y_1, \cdots, y_n, \cdots)$ the sequence of inventories, with initial inventory $y_1 = x$, we have the relation

$$y_{n+L} = y_{n+L-1} + v_n - D_{n+L-1}, \quad y_1 = x. \tag{13.2.1}$$

We note that we allow the possibility of backlog. Also the inventories $y_1, \cdots, y_L$ do not depend of the control, instead of just y_1. It is convenient to introduce pending orders $x_1, \cdots, x_{L-1}$ to be received at time $2, \cdots, L$, if $L \geq 2$. They are given values, like the initial value x. Therefore we have the relations

$$y_j = x + x_1 + \cdots + x_{j-1} - D_1 - \cdots - D_{j-1},\ 2 \leq j \leq L, \tag{13.2.2}$$

for $L \geq$2. We next define the costs per period

$$C(v) = K\mathbb{1}_{v>0} + cv, \quad l(x) = hx^+ + px^-. \tag{13.2.3}$$

Introduce the vector in R^L, $X = (x, x_1, \cdots, x_{L-1})$. We can define the objective functional

$$J_X(V) = \sum_{n=1}^{\infty} \alpha^{n-1} EC(v_n) + \sum_{j=1}^{L-1} \alpha^{j-1} El(y_j) + \sum_{n=1}^{\infty} \alpha^{n-2+L} El(y_{n+L-1}). \tag{13.2.4}$$

If $L = 1$ the intermediary sum disappears and we recover the standard cost objective. This intermediary sum is a given function of X, not depending on the control. The rest depends on X through y_L, so in fact depends on X only through the number

$$z = x + x_1 + \cdots + x_{L-1}. \tag{13.2.5}$$

Therefore we can write

$$J_X(V) = \sum_{j=1}^{L-1} \alpha^{j-1} El(y_j) + \tilde{J}_z(V),$$

where

$$\tilde{J}_z(V) = \sum_{n=1}^{\infty} \alpha^{n-1} EC(v_n) + \sum_{n=1}^{\infty} \alpha^{n-2+L} El(y_{n+L-1}). \tag{13.2.6}$$

Note that z represents the initial inventory augmented by the sum of the pending deliveries. It is called the *inventory position.* Similarly, we use for $L \geq 2$

$$y_{n+L-1} = y_n + v_{n-1} + \cdots + v_{n-L+1} - (D_n + \cdots + D_{n+L-2}),$$

with the notation

$$v_0 = x_{L-1}; v_{-1} = x_{L-2} \cdots ; v_{2-L} = x_1,$$

to define

$$\begin{aligned} z_n &= y_{n+L-1} + D_n + \cdots + D_{n+L-2} \\ &= y_n + v_{n-1} + \cdots + v_{n-L+1}, \end{aligned}$$

which is the inventory position at time n. We have $z_1 = z$. We then have the evolution

$$z_{n+1} = z_n + v_n - D_n. \tag{13.2.7}$$

This proves that the process z_n is adapted to the filtration $\mathcal{F}^{n-1}$. Next

$$l(y_{n+L-1}) = l(z_n - (D_n + \cdots + D_{n+L-2})).$$

But $D_n, \cdots, D_{n+L-2}$ are independent from z_n. Therefore

$$\alpha^{L-1} El(y_{n+L-1}) = E\tilde{l}(z_n),$$

where

$$\tilde{l}(x) = \alpha^{L-1} El\left(x - \sum_{j=1}^{L-1} D_j\right). \tag{13.2.8}$$

We can then write

$$\tilde{J}_z(V) = \sum_{n=1}^{\infty} \alpha^{n-1} EC(v_n) + \sum_{n=1}^{\infty} \alpha^{n-1} E\tilde{l}(z_n). \tag{13.2.9}$$

The problem has been reduced to a standard one, replacing the inventory with the inventory position, and the function $l(x)$ with $\tilde{l}(x)$.

Remark 13.1. The preceding transformation fails when backlog is not permitted. Dynamic Programming remains possible, but we have to work in R^L.

13.2.2. ANALYTICAL PROOF. In the preceding section, we have made a reasoning based on the control itself and show how it can be reduced to a one dimensional problem for the inventory position. We give here an analytical proof, following Karlin and Scarf, see [**1**]. The idea is to follow Remark 13.1 and to write the Bellman equation in the space R^L. The state at time n is a vector

$$Y_n = (y_n, y_n^1, \cdots, y_n^{L-1}).$$

The interpretation is easy. The first component y_n is the stock on hand and $y_n^1, \cdots, y_n^{L-1}$ represent the pending orders in the order of arrival; y_n^1 will arrive at time $n+1$ and y_n^{L-1} will arrive at time $n+L-1$. We have the initial condition

$$Y_1 = (x, x_1, \cdots, x_{L-1}).$$

The evolution equations are as follows

$$\begin{gathered} y_{n+1} = y_n + y_n^1 - D_n; \\ y_{n+1}^1 = y_n^2; \\ \cdots \\ y_{n+1}^{L-2} = y_n^{L-1}; \\ y_{n+1}^{L-1} = v_n, \end{gathered} \tag{13.2.10}$$

and v_n to be decided at time n will arrive at time $n+L$. From (13.2.10), it is easy to deduce (13.2.1). The cost functional is defined by

$$J_X(V) = \sum_{n=1}^{\infty} \alpha^{n-1}(C(v_n) + l(y_n)). \tag{13.2.11}$$

Call

$$w(X) = w(x, x_1, \cdots, x_{L-1}) = \inf_V J_X(V). \tag{13.2.12}$$

Bellman equation reads

$$w(x, x_1, \cdots, x_{L-1}) = l(x) + \inf_{v \geq 0}[C(v) + \alpha E w(x + x_1 - D, x_2, \cdots, x_{L-1}, v)]. \tag{13.2.13}$$

If we consider the Bellman equation relative to the inventory position, for the value function of $\tilde{J}_z(V)$,

$$u(z) = \inf_V \tilde{J}_z(V), \tag{13.2.14}$$

then we have

$$u(z) = \tilde{l}(z) + \inf_{v \geq 0}[C(v) + \alpha E u(z + v - D)]. \tag{13.2.15}$$

Proposition 13.1. *One has the formula*

$$w(x, x_1, \cdots, x_{L-1}) = u(x + x_1 + \cdots + x_{L-1}) + G(x, x_1, \cdots, x_{L-2}), \tag{13.2.16}$$

with

$$G(x, x_1, \cdots, x_{L-2}) = l(x) + \sum_{j=1}^{L-2} \alpha^j E l(x + x_1 + \cdots + x_j - (D_1 + \cdots + D_{L-1})). \tag{13.2.17}$$

PROOF. If one inserts (13.2.16) in (13.2.13), with $u(z)$ solution of (13.2.15), then we obtain a functional relation for G, namely

$$\tilde{l}(x) + G(x, x_1, \cdots, x_{L-2}) = l(x) + \alpha E G(x + x_1 - D, x_2, \cdots, x_{L-1}),$$

and by direct checking, we see that G defined by (13.2.17), satisfies the desired relation. □

13.2.3. MULTI-ECHELON INVENTORY PROBLEM. This problem was introduced by the seminal paper of Clark and Scarf, see [**16**]. We will consider $L = 2$. So we have

$$\begin{aligned} y_{n+1} &= y_n + y_n^1 - D_n; \\ y_{n+1}^1 &= v_n. \end{aligned} \tag{13.2.18}$$

However, the order v_n is not simply made on the market. It is obtained from an upper level installation, called installation 2, whereas the installation directly in contact with customers is called installation 1. Only installation 2 buys on the market. So we introduce the stock at installation 2, denoted by η_n and its evolution is defined by

$$\eta_{n+1} = \eta_n + \tilde{v}_n - v_n, \tag{13.2.19}$$

where $\tilde{v}_n$ is bought externally. We need to impose the constraint

$$0 \leq v_n \leq \eta_n, \tag{13.2.20}$$

which expresses the fact that the quantity v_n must be available at installation 2 to be ordered by installation 1. Note that the stock $\eta_n \geq 0$. It is convenient to introduce the quantity

$$\xi_n = \eta_n + y_n + y_n^1,$$

called by Clark-Scarf the system stock. It clearly satisfies the evolution equation

$$\xi_{n+1} = \xi_n + \tilde{v}_n - D_n. \tag{13.2.21}$$

The costs per period involve $C(v_n)$, $C_1(\tilde{v}_n)$ and costs $l(y_n)$, $l_1(\xi_n)$. Of course one could expect costs related to η_n and possibly to y_n^1. The fact that these costs are summarized in a cost depending only of the system stock is a restriction. This assumption will be satisfied in the case when the holding costs are identical for y_n, y_n^1, η_n and linear. This is thanks to the fact that $y_n^1, \eta_n \geq 0$. Define the control policy

$$V = \{v_1, \tilde{v}_1, \cdots, v_n, \tilde{v}_n, \cdots\},$$

and the control policy is adapted to the filtration $\mathcal{F}^{n-1}$. Recall the constraint (13.2.20), which can be expressed as

$$0 \leq v_n \leq \xi_n - y_n - y_n^1. \tag{13.2.22}$$

Denote by $\{x, x_1, \xi\}$ the initial state. We can define the cost objective

$$J_{x,x_1,\xi}(V) = \sum_{n=1}^{\infty} \alpha^{n-1}[C(v_n) + C_1(\tilde{v}_n) + l(y_n) + l_1(\xi_n)], \tag{13.2.23}$$

and the value function

$$\Psi(x, x_1, \xi) = \inf_V J_{x,x_1,\xi}(V), \tag{13.2.24}$$

then the value function is the solution of Bellman equation

$$\Psi(x,x_1,\xi) = l(x) + l_1(\xi) + \inf_{\substack{\tilde{v}\geq 0\\ 0\leq v\leq \xi-(x+x_1)}} [C(v) + C_1(\tilde{v}) + \alpha E\Psi(x+x_1-D, v, \xi+\tilde{v}-D)], \tag{13.2.25}$$

and the arguments satisfy $x_1 \geq 0, x+x_1 \leq \xi$. We next consider the problem of the previous section with $L=2$, equation (13.2.13) namely $w(x,x_1)$ solution of

$$w(x,x_1) = l(x) + \inf_{v\geq 0}[C(v) + \alpha E w(x+x_1-D, v)]. \tag{13.2.26}$$

We know from Proposition 13.1 that

$$w(x,x_1) = l(x) + u(x+x_1), \tag{13.2.27}$$

with

$$u(z) = \tilde{l}(z) + \inf_{v\geq 0}[C(v) + \alpha E u(z+v-D)], \tag{13.2.28}$$

and $\tilde{l}(z) = \alpha E l(z-D)$. We shall assume

$$C(v) = cv, \tag{13.2.29}$$

so we exclude the situation of set up cost. In that case, the function $u(z)$ is convex and the optimal feedback is given by a base stock policy, S. Define

$$G(\eta) = c\eta + \alpha E u(\eta - D),$$

then one has

$$\inf_{\eta\geq z} G(\eta) = \begin{cases} G(S), & \text{if } z \leq S \\ G(z), & \text{if } z \geq S \end{cases} \tag{13.2.30}$$

and from (13.2.28), we can write

$$u(z) = \tilde{l}(z) - cz + \begin{cases} G(S), & \text{if } z \leq S \\ G(z), & \text{if } z \geq S \end{cases} \tag{13.2.31}$$

We can then state the

Theorem 13.1. *For the problem* (13.2.18), (13.2.21), (13.2.22), (13.2.23) *with assumption* (13.2.29) *the value function* $\Psi(x,x_1,\xi)$ *solution of Bellman equation* (13.2.25) *is given by*

$$\Psi(x,x_1,\xi) = l(x) + u(x+x_1) + \Phi(\xi), \tag{13.2.32}$$

where $\Phi(\xi)$ *is the solution of Bellman equation*

$$\Phi(\xi) = l_1(\xi) + (G(\xi) - G(S))1\!\!1_{\xi\leq S} + \inf_{\tilde{v}\geq 0}[C_1(\tilde{v}) + \alpha E\Phi(\xi+\tilde{v}-D)]. \tag{13.2.33}$$

If one denotes by $\hat{\tilde{v}}(\xi)$ *the optimal feedback defined by equation* (13.2.33), *then the optimal feedback defined by the full Bellman equation* (13.2.25) *is given by*

$$\hat{v}(x,x_1,\xi) = \begin{cases} \xi - (x+x_1), & \textit{if } x+x_1 \leq \xi \leq S \\ S-(x+x_1), & \textit{if } x+x_1 \leq S \leq \xi \\ 0, & \textit{if } S \leq x+x_1 \leq \xi \end{cases} \tag{13.2.34}$$

and

$$\hat{\tilde{v}}(x,x_1,\xi) = \hat{\tilde{v}}(\xi). \tag{13.2.35}$$

PROOF. By direct checking, we see that a function of the form (13.2.32) will satisfy (13.2.25) if

$$\Phi(\xi) + \inf_{x+x_1 \leq \eta} G(\eta) = l_1(\xi) + \inf_{x+x_1 \leq \eta \leq \xi} G(\eta) + \inf_{\tilde{v} \geq 0}[C_1(\tilde{v}) + \alpha E \Phi(\xi + \tilde{v} - D)].$$

However, it is easy to check that

$$\inf_{x+x_1 \leq \eta \leq \xi} G(\eta) - \inf_{x+x_1 \leq \eta} G(\eta) = \begin{cases} 0, & \text{if } \quad \xi \geq S \\ G(\xi) - G(S), & \text{if } \quad \xi \leq S \end{cases}$$

The important property is that this expression does not depend on $x + x_1$. Then $\Phi(\xi)$ must be solution of (13.2.33). It also follows that the expression on the right hand side of (13.2.32) is solution of (13.2.25) and the feedback defined by (13.2.34), (13.2.35) is optimal. This completes the proof. □

13.2.4. RANDOM LEAD TIME. We assume now that the lead time at time n when the order v_n is decided is a random variable L_n. We assume the restrictive assumption that

(13.2.36)
$$\begin{aligned} L_{n+1} &= L_n + \epsilon_n, \quad \epsilon_n \geq 0, \text{ i.i.d} \\ L_1 &\geq 1 \quad \text{independent of } \epsilon_n \end{aligned}$$

So the lead time cannot improve with time. It contains of course the deterministic case, and more generally the case of a lead time which is constant, but random. It can be considered as a risk averse attitude towards lead times, but of course eliminates any possibility of improvement.

We also assume

(13.2.37) $\quad L_n \quad$ independent of the demand process.

We then write

(13.2.38)
$$y_{n+L_n} = y_{n+L_n-1} + v_n - D_{n+L_n-1}, \quad y_1 = x,$$

and $n + L_n$ is a random time. We define again the inventory position

(13.2.39)
$$z_{n+1} = y_{n+L_n} + D_{n+1} + \cdots + D_{n+L_n-1}.$$

If $L_n = 1$, we interpret the formula as $z_{n+1} = y_{n+1}$. We note that

$$y_{n+L_n} = y_{n-1+L_{n-1}} + v_n - (D_{n-1+L_{n-1}} + \cdots + D_{n+L_n-1}),$$

since only v_n arrives in between the times $n - 1 + L_{n-1}$ (excluded) and $n + L_n$. Therefore we have again

(13.2.40)
$$z_{n+1} = z_n + v_n - D_n.$$

The process z_n is adapted to $\mathcal{F}^{n-1}$. To define z_1, we have to take into account the pending orders at time 1. This means that we have to consider v_n with $n \leq 0$. We write $v_{-k}, k \geq 0$. Since we must have

$$L_{-k} - k \geq 2,$$

to have the order v_{-k} pending at time 1, there can be only a finite number of them, although this number k^* can be random. We then have

$$z_1 = x + \sum_{k=0}^{k^*} v_{-k},$$

and we must have

$$L_{-k+1} \geq L_{-k} \geq L_{-k^*} \geq k^* + 2.$$

It follows that

$$z_n = y_n + \sum_{\substack{-k^* \leq k \leq n-1 \\ k + L_k \geq n+1}} v_k.$$

Of course, the orders v_{-k} are given. So it is sufficient to give the inventory position at time 1, namely $z_1 = z$. Consider the cost objective

$$J(V) = \sum_{j=1}^{\infty} \alpha^{j-1} EC(v_j) + \sum_{j=1}^{\infty} \alpha^{j-1} El(y_j).$$

From our assumption on the lead times, the function $n + L_n$ is strictly monotone increasing in k. So we can write

$$\begin{aligned} J(V) &= \sum_{j=1}^{\infty} \alpha^{j-1} EC(v_j) + E \sum_{j=1}^{L_0 - 1} \alpha^{j-1} El(y_j) \\ &\quad + \sum_{n=1}^{\infty} E \sum_{n+L_{n-1}-1}^{n+L_n-1} \alpha^{j-1} El(y_j) \end{aligned}$$

Next

$$E \sum_{n+L_{n-1}-1}^{n+L_n-1} \alpha^{j-1} El(y_j) = E\alpha^{n+L_{n-1}-2} \sum_{k=0}^{L_n - L_{n-1}} \alpha^k l(y_{n-1+L_{n-1}+k}).$$

However

$$y_{n-1+L_{n-1}+k} = y_{n-1+L_{n-1}} - (D_{n-1+L_{n-1}} + \cdots + D_{n-2+L_{n-1}+k}), \quad 0 \leq k \leq L_n - L_{n-1},$$

where the sum is omitted if $k = 0$. This follows from the fact that there is no order arriving at time $n - 1 + L_{n-1} + k$, $1 \leq k \leq L_n - L_{n-1}$. Taking account of the value of $y_{n-1+L_{n-1}}$, we get

$$y_{n-1+L_{n-1}+k} = z_n - (D_n + \cdots + D_{n-2+L_{n-1}+k}), \quad 0 \leq k \leq L_n - L_{n-1}.$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} E \sum_{n+L_{n-1}-1}^{n+L_n-1} \alpha^{j-1} El(y_j) &= \sum_{n=1}^{\infty} E\alpha^{n+L_{n-1}-2} \\ &\quad \cdot \sum_{k=0}^{L_n - L_{n-1}} \alpha^k l(z_n - (D_n + \cdots + D_{n+L_{n-1}+k-2})). \end{aligned}$$

Define the function

$$\lambda(i, x) = El(x - (D_1 + \cdots + D_i)), \ i \geq 0, \tag{13.2.41}$$

in which the sum is omitted for $i = 0$. Hence, $\lambda(0, x) = l(x)$.

From the independence properties, one can write

$$\sum_{n=1}^{\infty} E \sum_{n+L_{n-1}-1}^{n+L_n-1} \alpha^{j-1} El(y_j) = \sum_{n=1}^{\infty} \alpha^{n-1} E\alpha^{L_{n-1}-1} \sum_{k=0}^{\epsilon_{n-1}} \alpha^k \lambda(L_{n-1} - 1 + k, z_n)$$

Define next

$$\tilde{l}_n(x) = E\alpha^{L_{n-1}-1}\sum_{k=0}^{\epsilon_{n-1}} \alpha^k \lambda(L_{n-1} - 1 + k, x). \tag{13.2.42}$$

From the independence of the demands and the lead times, we obtain

$$\sum_{n=1}^{\infty} E \sum_{n+L_{n-1}-1}^{n+L_n-1} \alpha^{j-1} El(y_j) = \sum_{n=1}^{\infty} \alpha^{n-1} E\tilde{l}_n(z_n).$$

Next the quantity $E\sum_{j=1}^{L_0-1} \alpha^{j-1}El(y_j)$ is fixed, independent of the control. So we are finally led to the following optimization problem

$$\tilde{J}_z(V) = \sum_{n=1}^{\infty} \alpha^{n-1} EC(v_n) + \sum_{n=1}^{\infty} \alpha^{n-1} E\tilde{l}_n(z_n), \tag{13.2.43}$$

with the evolution (13.2.40). This problem is not standard, because it is non stationary. The cost on each period n depends on n. We have not treated non stationary problems in this book, to simplify notation. However, the techniques of Dynamic Programming adapt to non stationary problems in infinite horizon. Unlike in the finite horizon case, it does not reduce to a sequence. It remains a functional equation, since one cannot initiate the sequence.

If $L_n = L$ (random), then $\epsilon_n = 0$. In this case

$$\tilde{l}_n(x) = \tilde{l}(x) = E\alpha^{L-1}\lambda(L-1, x),$$

and if L is deterministic we recover (13.2.8).

13.3. MODELS WITHOUT INVENTORY POSITION

13.3.1. CONSTRAINT ON THE ORDERING. We consider the model of section 13.2.1, in which the lead time L is deterministic. However, we will introduce a constraint in the ordering process. One cannot order when there is a pending order. Therefore, the number of pending orders is at most one. An ordering policy V is composed of a sequence of ordering times

$$\tau_1, \cdots, \tau_j, \cdots,$$

which are $\mathcal{F}^{n-1}$ stopping times. To each ordering time τ_j, one associates a quantity v_{τ_j} which is $\mathcal{F}^{\tau_j - 1}$ measurable, which is the amount ordered. The new element is that the usual condition $\tau_j \leq \tau_{j+1}$ is replaced with the constraint

$$\tau_{j+1} \geq \tau_j + L.$$

The inventory evolves as follows

$$y_{n+1} = y_n + v_n - D_n, \quad y_1 = x,$$

with

$$v_n = 0, \text{ if } n \neq \tau_j, \text{ for some } j.$$

The objective function is the same as usual

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1}(l(y_n) + C(v_n)),$$

and the value function is

$$u(x) = \inf_V J_x(V).$$

13.3.2. DYNAMIC PROGRAMMING. It is easy to figure out the Bellman equation, considering the possibilities at the initial time. If there is no order at time 0, then the best which can be achieved is

$$l(x) + \alpha E u(x - D).$$

On the other hand if an order of size v is put at time 0, which is possible since there is no pending order, then the inventory evolves as follows

$$\begin{aligned} y_1 &= x; \\ y_2 &= x - D_1; \\ &\cdots \\ y_{L+1} &= x - (D_1 + \cdots + D_L) + v, \end{aligned}$$

and the best which can be achieved is

$$K + cv + l(x) + E\sum_{j=2}^{L} \alpha^{j-1} l(x - (D_1 + \cdots + D_{j-1})) + \alpha^L E u(x + v - (D_1 + \cdots + D_L)).$$

From these considerations we can easily derive Bellman equation

$$u(x) = \min\left\{ l(x) + \alpha E u(x - D), \right. \tag{13.3.1}$$

$$\left. K - cx + E\sum_{j=0}^{L-1} \alpha^j l(x - D^{(j)}) + \inf_{\eta > x} [c\eta + \alpha^L E u(\eta - D^{(L)})] \right\}$$

in which we use the notation

$$D^{(j)} = D_1 + \cdots + D_j \quad \text{if } j \geq 1, \ \text{if } j = 0$$

13.3.3. s, S **POLICY.** We first transform equation (13.3.1), by introducing a constant $s \in R$, and changing $u(x)$ into $H_s(x)$, using the formula

$$u(x) = -cx + E\sum_{j=0}^{L-1} \alpha^j l(x - D^{(j)}) + C_s + H_s(x), \tag{13.3.2}$$

where C_s will be defined below. In fact, let

$$g(x) = (1 - \alpha)cx + \alpha^L E l(x - D^{(L)}),$$

then we take

$$C_s = \frac{g(s) + \alpha c\bar{D}}{1 - \alpha},$$

where $\bar{D}$ is the expected value of D.

We then note the formula

$$E\sum_{j=0}^{L-1} \alpha^j g(x - D^{(j)}) = cx(1 - \alpha^L) - \alpha c\bar{D}\frac{1 - \alpha^L}{1 - \alpha} + \alpha^L c\bar{D}L + E\sum_{j=0}^{L-1} \alpha^{j+L} l(x - D^{(j+L)})$$

We then check that $H_s(x)$ is the solution of

$$H_s(x) = \min\left\{ g(x) - g(s) + \alpha E H_s(x - D), \right. \tag{13.3.3}$$

$$\left. K + \inf_{\eta > x}\left[E\sum_{j=0}^{L-1} \alpha^j (g(\eta - D^{(j)}) - g(s)) + \alpha^L E H_s(\eta - D^{(L)}) \right] \right\}.$$

On the other hand, for any s we define in a unique way the solution of

$$\begin{aligned} H_s(x) &= g(x) - g(s) + \alpha E H_s(x - D),\ x \geq s \\ &= 0,\ x \leq s \end{aligned} \tag{13.3.4}$$

We have used the same notation $H_s(x)$ between (13.3.3) and (13.3.4), because we want to find s so that the solution of (13.3.4) is also a solution of (13.3.3). Calling $\mu(x) = g'(x)$, the solution of (13.3.4) is explicitly given by

$$H_s(x) = \int_s^x \Gamma(x - \eta) f(\eta) d\eta,\ x > s, \tag{13.3.5}$$

where we recall $\Gamma(x)$ is the solution of

$$\Gamma(x) = 1 + \alpha \int_0^x \Gamma(x - \eta) f(\eta) d\eta,\ x > 0. \tag{13.3.6}$$

Let us set

$$g_s(x) = (g(x) - g(s)) 1\!\!1_{x \geq s},$$

then the solution of (13.3.4) satisfies

$$H_s(x) = g_s(x) + \alpha E H_s(x - D),\ \forall x. \tag{13.3.7}$$

From this relation we deduce, after iterating that

$$H_s(x) = E \sum_{j=0}^{L-1} \alpha^j g_s(x - D^{(j)}) + \alpha^L E H_s(x - D^{(L)}),$$

therefore also

$$\begin{aligned} &E \sum_{j=0}^{L-1} \alpha^j (g(x - D^{(j)}) - g(s)) + \alpha^L E H_s(x - D^{(L)}) \\ &\qquad = H_s(x) + E \sum_{j=0}^{L-1} \alpha^j (g(x - D^{(j)}) - g(s)) 1\!\!1_{x - D^{(j)} < s}. \end{aligned}$$

Define

$$\Psi_s(x) = H_s(x) + E \sum_{j=0}^{L-1} \alpha^j (g(x - D^{(j)}) - g(s)) 1\!\!1_{x - D^{(j)} < s}, \tag{13.3.8}$$

then if we want $H_s(x)$ to satisfy (13.3.3), we must have

$$H_s(x) = \min \left\{ g(x) - g(s) + \alpha E H_s(x - D), K + \inf_{\eta > x} \Psi_s(\eta) \right\}. \tag{13.3.9}$$

This relation motivates the choice of s. We want to find s so that

$$K + \inf_{\eta > s} \Psi_s(\eta) = 0. \tag{13.3.10}$$

If such an s exists, we define $S = S(s)$ as the point where the infimum is attained.

$$\inf_{\eta > s} \Psi_s(\eta) = \Psi_s(S(s)). \tag{13.3.11}$$

Note that $\Psi_s(x)$ coincides with $H_s(x)$, for $x > s$, when $L = 1$.

13.3.4. EXISTENCE AND CHARACTERIZATION OF THE PAIR s, S. We recall that

$$\mu(x) = (1-\alpha)c + \alpha^L h - \alpha^L (h+p)\bar{F}^{(L)}(x), \tag{13.3.12}$$

where $F^{(L)}(x)$ is the cumulative distribution function of $D^{(L)}$ and $\bar{F}^{(L)}(x) = 1 - F^{(L)}(x)$. We assume that

$$(1-\alpha)c - \alpha^L p < 0. \tag{13.3.13}$$

There exists a unique $\bar{s} > 0$ such that

$$\mu(x) \leq 0, \text{ if } x \leq \bar{s}, \ \mu(x) \geq 0, \text{ if } x \geq \bar{s}.$$

We perform next further computations on the function $\Psi(x)$. We have

$$\begin{aligned}
&E[(g(x-D^{(j)}) - g(s))\mathbb{1}_{x-D^{(j)}<s}] \\
&= -E\mathbb{1}_{D^{(j)}>x-s}\left(\int_{x-D^{(j)}}^{s} \mu(\xi)d\xi\right) \\
&= -\int_{x-s}^{+\infty} dF^{(j)}(\zeta) \int_{x-\zeta}^{s} \mu(\xi)d\xi \\
&= -\int_{x-s}^{+\infty} \bar{F}^{(j)}(\zeta)\mu(x-\zeta)d\zeta \\
&= -\int_{-\infty}^{s} \bar{F}^{(j)}(x-\xi)\mu(\xi)d\xi \\
&= -\int_{-\infty}^{x} \bar{F}^{(j)}(x-\xi)\mu(\xi)d\xi + \int_{s}^{x} \bar{F}^{(j)}(x-\xi)\mu(\xi)d\xi.
\end{aligned}$$

We then check that

$$\begin{aligned}
&H_s(x) + \sum_{j=0}^{L-1} \alpha^j \int_s^x \bar{F}^{(j)}(x-\xi)\mu(\xi)d\xi \\
&\quad = \frac{1-\alpha^L}{1-\alpha}(g(x) - g(s)) + \sum_{j=L}^{\infty} \alpha^j \int_s^x F^{(j)}(x-\xi)\mu(\xi)d\xi.
\end{aligned}$$

This formula holds for any x, and the integral term vanishes when $x < s$. Collecting results, one obtains the formula

$$\begin{aligned}
\Psi_s(x) \quad = \quad & \frac{1-\alpha^L}{1-\alpha}(g(x) - g(s)) + \sum_{j=L}^{\infty} \alpha^j \int_s^x F^{(j)}(x-\xi)\mu(\xi)d\xi \\
& -((1-\alpha)c + \alpha^L h)\bar{D} \sum_{j=1}^{L-1} j\alpha^j \\
& +\alpha^L (h+p) \sum_{j=1}^{L-1} \alpha^j \int_0^{+\infty} \bar{F}^{(j)}(\zeta)\bar{F}^{(L)}(x-\zeta)d\zeta.
\end{aligned} \tag{13.3.14}$$

It follows that $\Psi_s(x)$ is bounded below and tend to $+\infty$ as $x \to +\infty$. Therefore the infimum over $x \geq s$ is attained and we can define $S(s)$ as the smallest infimum.

Proposition 13.2. *We assume* (13.3.13). *One has*

$$\max_s \Psi_s(S(s)) = \Psi_{\bar{s}}(S(\bar{s})) \geq 0, \tag{13.3.15}$$

$$\Psi_s(S(s)) \to -\infty, \ as\ s \to -\infty \tag{13.3.16}$$

$$\Psi_s(S(s)) \to -((1-\alpha)c + \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j, \ as\ s \to +\infty. \tag{13.3.17}$$

There exists one and only one solution of (13.3.10) *such that* $s \leq \bar{s}$.

PROOF. We compute

$$\Psi_s'(x) = \frac{1-\alpha^L}{1-\alpha}\mu(x) + \sum_{j=L}^{\infty}\alpha^j\int_s^x f^{(j)}(x-\xi)\mu(\xi)d\xi$$
$$-\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j\int_0^x \bar{F}^{(j)}(\zeta)f^{(L)}(x-\zeta)d\zeta$$

which can also be written as

$$\Psi_s'(x) = \gamma(x) + \sum_{j=L}^{\infty}\alpha^j\int_s^x f^{(j)}(x-\xi)\mu(\xi)d\xi, \tag{13.3.18}$$

with

$$\gamma(x) = \frac{1-\alpha^L}{1-\alpha}((1-\alpha)c - \alpha^L p) + \alpha^L(h+p)\sum_{j=0}^{L-1}\alpha^j\int_0^x F^{(j)}(x-\xi)f^{(L)}(\xi)d\xi. \tag{13.3.19}$$

The function $\gamma(x)$ is increasing and

$$\gamma(x) = \frac{1-\alpha^L}{1-\alpha}((1-\alpha)c - \alpha^L p) < 0, \forall x \leq 0, \ \gamma(+\infty) = \frac{1-\alpha^L}{1-\alpha}((1-\alpha)c + \alpha^L h),$$

therefore there exists a unique s^* such that

$$\gamma(x) < 0, \forall x < s^*, \ \gamma(x) > 0, \forall x > s^*, \gamma(s^*) = 0, \ s^* > 0.$$

Note that

$$\mu(x)\frac{1-\alpha^L}{1-\alpha} - \gamma(x) = \alpha^L(h+p)\sum_{j=0}^{L-1}\alpha^j\int_0^x \bar{F}^{(j)}(x-\xi)f^{(L)}(\xi)d\xi.$$

This quantity vanishes for $x \leq 0$ or $L = 1$. Otherwise

$$\mu(x)\frac{1-\alpha^L}{1-\alpha} - \gamma(x) > 0, \forall x > 0, \ L \geq 2$$

Since $\gamma(s^*) = 0$, we have $\mu(s^*) > 0$, hence $0 < \bar{s} < s^*$.

Next, if $s \geq s^*$, $\Psi_s'(x) \geq 0, \forall x \geq s$, hence $S(s) = s$. Therefore

$$\Psi_s(S(s)) = -((1-\alpha)c + \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j$$
$$+\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j\int_0^{+\infty}\bar{F}^{(j)}(\zeta)\bar{F}^{(L)}(s-\zeta)d\zeta$$

this function decreases in s and converges to $-((1-\alpha)c+\alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j$ as $s\to+\infty$. Consider now $\bar{s}<s<s^*$. We note that

$$\begin{aligned}\Psi_s'(s) &= \gamma(s)<0,\\ \Psi_s'(s^*) &= \sum_{j=L}^{\infty}\alpha^j\int_s^{s^*} f^{(j)}(s^*-\xi)\mu(\xi)d\xi>0,\end{aligned}$$

hence in this case

$$\bar{s}<s<S(s)<s^*.$$

If $s<\bar{s}$, then we can claim that

$$s<\bar{s}<S(s).$$

Indeed, for $s<x<\bar{s}$, we can see from formula (13.3.18) that $\Psi_s'(x)<0$. However, we cannot compare $S(s)$ with s^* in this case.

We then study the behavior of $\Psi_s(S(s))$. We have already seen that, for $s>s^*$ the function $\Psi_s(S(s))$ is decreasing to the negative constant

$$-((1-\alpha)c+\alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j.$$

In this case

$$\frac{d}{ds}\Psi_s(S(s)) = -\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j\int_0^s \bar{F}^{(j)}(\zeta)f^{(L)}(s-\zeta)d\zeta,\ s>s^*. \tag{13.3.20}$$

Note that $s>0$. If $s<s^*$, then $s<S(s)$, therefore

$$\frac{d}{ds}\Psi_s(S(s)) = \frac{\partial\Psi_s}{\partial s}(S(s)),$$

hence

$$\frac{d}{ds}\Psi_s(S(s)) = -\mu(s)\left[\frac{1-\alpha^L}{1-\alpha}+\sum_{j=L}^{\infty}\alpha^j F^{(j)}(S(s)-s)\right]. \tag{13.3.21}$$

Note that at $s=s^*$ the two formulas coincide, since

$$-\mu(s^*)\frac{1-\alpha^L}{1-\alpha} = -\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j\int_0^{s^*}\bar{F}^{(j)}(\zeta)f^{(L)}(s^*-\zeta)d\zeta,$$

which can be easily checked, by remembering the definition of $s^*(\gamma(s^*)=0)$, and simple calculations. It follows clearly that $\Psi_s(S(s))$ decreases on $(\bar{s},s^*)$ and increases on $(-\infty,\bar{s})$. Finally $\Psi_s(S(s))$ decreases on $(\bar{s},+\infty)$ and increases on $(-\infty,\bar{s})$. So it attains its maximum at $\bar{s}$. From the formula for $\Psi_{\bar{s}}(x)$, $H_{\bar{s}}(x)$, $x>\bar{s}$ and the fact that $\bar{s}$ is the minimum of g, we get immediately that $\Psi_{\bar{s}}(S(\bar{s}))\geq 0$. Finally, for $s<0$ we have

$$\begin{aligned}\Psi_s(S(s)) &\leq \Psi_s(0)\\ &= ((1-\alpha)c-\alpha^L p)\left[\frac{1-\alpha^L}{1-\alpha}(-s)+\sum_{j=L}^{\infty}\alpha^j\int_s^0 F^{(j)}(-\xi)d\xi-\bar{D}\sum_{j=1}^{L-1} j\alpha^j\right],\end{aligned}$$

which tends to $-\infty$ as $s \to -\infty$. Therefore $\Psi_s(S(s))$ is increasing from $-\infty$ to a positive number as s grows from $-\infty$ to $\bar{s}$. Therefore there is one and only one $s < \bar{s}$ satisfying (13.3.10). The proof has been completed. □

13.3.5. SOLUTION AS AN s, S POLICY. It remains to see whether the function $H_s(x)$ defined by (13.3.5) with s solution of (13.3.10) is indeed a solution of equation (13.3.9). It is useful to use, instead of $\Psi_s(x)$ a different function, namely

$$\Phi_s(x) = H_s(x) + E\sum_{j=1}^{L-1} \alpha^j (g(x - D^{(j)}) - g(s))\mathbb{1}_{x-D^{(j)}<s}, \tag{13.3.22}$$

which differs from $\Psi_s(x)$ by simply deleting the term corresponding to $j = 0$. Clearly

$$\Phi_s(x) = \Psi_s(x), \forall x \geq s.$$

However, when $x < s$, the two functions differ by the term

$$\Psi_s(x) = \Phi_s(x) + g(x) - g(s), \ \forall x < s.$$

Note that in finding S it is indifferent to work with one or the other function. Note also that, when $L = 1$, $\Phi_s(x) = H_s(x), \forall x$. Now we have

$$\inf_{\eta > x} \Psi_s(\eta) = \inf_{\eta > x} \Phi_s(\eta).$$

This equality is obvious when $x > s$, since the functions are identical. If $x < s$ we have

$$\inf_{\eta > x} \Psi_s(\eta) = \inf_{\eta > x} \Phi_s(\eta) = \inf_{\eta > s} \Phi_s(\eta) = -K.$$

Indeed, for instance

$$\inf_{\eta > x} \Psi_s(\eta) = \min\left[\inf_{\eta > s} \Psi_s(\eta), \inf_{x<\eta<s} \Psi_s(\eta)\right],$$

and $\inf_{\eta > s} \Psi_s(\eta) = -K$, whereas

$$\inf_{x<\eta<s} \Psi_s(\eta) = \inf_{x<\eta<s} E\sum_{j=0}^{L-1} \alpha^j (g(\eta - D^{(j)}) - g(s))\mathbb{1}_{x-D^{(j)}<s} > 0,$$

hence the result. Therefore(13.3.9) becomes

$$H_s(x) = \min\left\{g(x) - g(s) + \alpha E H_s(x - D), K + \inf_{\eta > x} \Phi_s(\eta)\right\}. \tag{13.3.23}$$

For $x < s$, this relation reduces to

$$0 = \min[g(x) - g(s), 0]$$

which is true since $g(x)$ is decreasing for $x < s$. We then consider $x > s$. Since $H_s(x)$ is equal to the first term of the bracket. Therefore, what we have to prove is

$$H_s(x) \leq K + \inf_{\eta > x} \Phi_s(\eta), \forall x > s. \tag{13.3.24}$$

In fact, we shall not been able to prove (13.3.24) for all values of L. We know it is true for $L = 1$. We will prove it afterwards for Poisson demands. For general demand distributions we have the

Proposition 13.3. *We assume*

$$\alpha^L((1-\alpha)c - \alpha^L p)(L-1)\bar{D} + K(1-\alpha) \geq 0, \tag{13.3.25}$$

then the property (13.3.24) *is satisfied.*

PROOF. We note that this result includes the case $L = 1$ in which the condition is automatically satisfied. The proof is similar to that of the case $L = 1$.

We recall that

$$H_s(x) - \alpha E H_s(x - D) = g_s(x), \forall x. \tag{13.3.26}$$

We then find a similar equation for $\Phi_s(x)$. This is where it is important to consider $\Phi_s(x)$ and not $\Psi_s(x)$, since we write the equation for any x and not just for $x > s$. It is easy to verify that $\Phi_s(x)$ is the solution of

$$\begin{aligned}\Phi_s(x) - \alpha E \Phi_s(x - D) &= g_s(x) + \alpha E(g(x - D) - g(s))\mathbb{1}_{x-D<s} \\ &-\alpha^L E(g(x - D^{(L)}) - g(s))\mathbb{1}_{x-D^{(L)}<s},\end{aligned} \tag{13.3.27}$$

and again we check that this equation coincides with (13.3.26) when $L = 1$. Going back to (13.3.24) we recall that

$$K + \inf_{\eta > x} \Phi_s(\eta) \geq K + \inf_{\eta > s} \Phi_s(\eta) = 0, \forall x > s.$$

So it is sufficient to prove (13.3.24) for $x > x_0 > s$, the first value larger than s such that $H_s(x) \geq 0$. We necessarily have $H_s(x_0)$=0. we have $s < \bar{s} < x_0$. Let us fix $\xi > 0$. We consider the domain $x \geq x_0 - \xi$. We can write

$$H_s(x) - \alpha E H_s(x - D)\mathbb{1}_{x-D\geq x_0-\xi} \leq g_s(x), \forall x, \tag{13.3.28}$$

using the fact that

$$E H_s(x - D)\mathbb{1}_{x-D<x_0-\xi} \leq 0.$$

Define next

$$M_s(x) = \Phi_s(x + \xi) + K.$$

We note that $M_s(x) > 0, \forall x$. We then state that

$$\begin{aligned}M_s(x) - \alpha E M_s(x - D) &= g_s(x + \xi) + \alpha E(g(x + \xi - D) - g(s))\mathbb{1}_{x+\xi-D<s} \\ &-\alpha^L E(g(x + \xi - D^{(L)}) - g(s))\mathbb{1}_{x+\xi-D^{(L)}<s} + K(1-\alpha)\end{aligned}$$

and using the positivity of M we can assert that

$$\begin{aligned}M_s(x) - \alpha E M_s(x - D)\mathbb{1}_{x-D\geq x_0-\xi} &\geq g_s(x + \xi) \\ &+\alpha E(g(x + \xi - D) - g(s))\mathbb{1}_{x+\xi-D<s} \\ &-\alpha^L E(g(x + \xi - D^{(L)}) - g(s))\mathbb{1}_{x+\xi-D^{(L)}<s} + K(1-\alpha)\end{aligned} \tag{13.3.29}$$

We now consider the difference $Y_s(x) = H_s(x) - M_s(x)$, in the domain $x \geq x_0 - \xi$. We have

$$\begin{aligned}Y_s(x) - \alpha E Y_s(x - D)\mathbb{1}_{x-D\geq x_0-\xi} &\leq g_s(x) - g_s(x + \xi) \\ &-\alpha E(g(x + \xi - D) - g(s))\mathbb{1}_{x+\xi-D<s} \\ &+\alpha^L E(g(x + \xi - D^{(L)}) - g(s))\mathbb{1}_{x+\xi-D^{(L)}<s} - K(1-\alpha)\end{aligned} \tag{13.3.30}$$

We have seen from the case $L = 1$, Theorem 9.11, that

$$g_s(x + \xi) - g_s(x) \geq 0, \forall x \geq x_0 - \xi.$$

Consider next the function

$$\chi_s(y) = \alpha E(g(y - D) - g(s))\mathbb{1}_{y-D<s} - \alpha^L E(g(y - D^{(L)}) - g(s))\mathbb{1}_{y-D^{(L)}<s},$$

for $y \geq x_0$. We check that

$$\chi_s(y) = \int_{-\infty}^{s} d\zeta\mu(\zeta) \int_{y-\zeta}^{+\infty} (-\alpha f(\eta) + \alpha^L f^{(L)}(\eta))d\eta,$$

so in fact

$$\chi_s(y) = \int_{-\infty}^{s} \mu(\zeta)(-\alpha\bar{F}(y-\zeta) + \alpha^L \bar{F}^{(L)}(y-\zeta))d\zeta.$$

Note that in the integral, $\mu(\zeta) < 0$, since $s < \bar{s}$. We deduce

$$\chi_s(y) \geq \alpha^L \int_{-\infty}^{s} \mu(\zeta)(\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta))d\zeta.$$

Note that $\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta) \geq 0$, and $\mu(\zeta) \geq ((1-\alpha)c - \alpha^L p)$. Therefore

$$\chi_s(y) \geq \alpha^L((1-\alpha)c - \alpha^L p) \int_{-\infty}^{s} (\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta))d\zeta,$$

and since $y \geq s$,

$$\begin{aligned} \chi_s(y) &\geq \alpha^L((1-\alpha)c - \alpha^L p) \int_{-\infty}^{y} (\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta))d\zeta \\ &= \alpha^L((1-\alpha)c - \alpha^L p) \int_{0}^{\infty} (\bar{F}^{(L)}(u) - \bar{F}(u))du \\ &= \alpha^L((1-\alpha)c - \alpha^L p)(L-1)\bar{D} \end{aligned}$$

Thanks to the assumption (13.3.25) we can assert that

$$Y_s(x) - \alpha E Y_s(x-D) 1\!\!1_{x-D \geq x_0 - \xi} \leq 0, \ \forall x \geq x_0 - \xi.$$

Also

$$Y_s(x_0 - \xi) \leq -\Phi_s(x_0) - K \leq -K,$$

since $\Phi_s(x_0) \geq H_s(x_0) = 0$. It follows that

$$Y_s(x) \leq 0, \forall x \geq x_0 - \xi,$$

which is the desired result. □

13.3.6. POISSON DEMAND.

We state the

Proposition 13.4. *We assume that the demand is distributed according to a Poisson distribution, then* (13.3.24) *holds.*

PROOF. We are going to show that

$$H_s'(x) \geq 0, \text{ if } x \geq x_0, \tag{13.3.31}$$

then (13.3.24) will follow immediately, since for $x \geq x_0$

$$H_s(x) \leq H_s(x+\xi) \leq \Phi_s(x+\xi) \leq \Phi_s(x+\xi) + K, \forall \xi \geq 0$$

Assume

$$f(x) = \beta \exp -\beta x,$$

then, we know (see section 9.3.5) that

$$\Gamma(x) = \frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha} \exp -\beta(1-\alpha)x, \ x > 0,$$

and

$$\mu(x) = c(1-\alpha) + \alpha h - \alpha(p+h) \exp -\beta x^+.$$

We have

$$H_s'(x) = \mu(x) + \int_s^x \Gamma'(x-\xi)\mu(\xi)d\xi,$$

therefore

$$H_s'(x) = \mu(x) + \alpha\beta \exp -\beta(1-\alpha)x \int_s^x \exp \beta(1-\alpha)\xi\, \mu(\xi)d\xi,$$

and

$$H_s(x) = \int_s^x \frac{\mu(\xi)}{1-\alpha}d\xi - \frac{\alpha}{1-\alpha} \exp -\beta(1-\alpha)x \int_s^x \exp \beta(1-\alpha)\xi\, \mu(\xi)d\xi.$$

We recall that

$$H_s(x) \leq 0,\ \forall x < x_0,\ H_s(x_0) = 0.$$

Let now x_1 such that

$$\int_s^{x_1} \exp \beta(1-\alpha)\xi\, \mu(\xi)d\xi = 0.$$

If $x_1 \leq x_0$ then from the formula of $H_s'(x)$,$H_s'(x_0) > 0$ and the fact that $\mu(x)$ increases, we see immediately that $H_s'(x) \geq 0$, as $x \geq x_0$. So assume that $\bar{s} < x_0 < x_1$. For $x \geq x_1$

$$H_s'(x) \geq H_s'(x_1) \geq H_s'(x_0) > 0.$$

There remains the interval (x_0, x_1). We know that $H_s'(x_1) \geq H_s'(x_0) \geq 0$. If there exists a point in the interval such that $H_s'(x) < 0$, necessarily there exists a point ξ such that

$$x_0 < \xi < x_1,\quad H_s'(\xi) = 0.$$

If we introduce

$$\lambda(x) = \mu(x) \exp \beta(1-\alpha)x,\quad \rho(x) = \frac{\lambda(x)}{\int_s^x \lambda(u)du},$$

then ξ satisfies $\rho(\xi) = -\alpha\beta$. By definition of x_1,

$$\int_s^x \lambda(u)du < 0, \forall s < x < x_1,$$

and $\lambda'(x) > 0$. Therefore

$$\rho'(x) = \frac{\lambda'(x) \int_s^x \lambda(u)du - \lambda^2(x)}{(\int_s^x \lambda(u)du)^2} < 0.$$

Moreover

$$H_s'(x_0) > 0 \Rightarrow \lambda(x_0) + \alpha\beta \int_s^{x_0} \lambda(u)du > 0,$$

hence

$$\rho(x_0) + \alpha\beta < 0.$$

Therefore $\rho(x) < -\alpha\beta, \forall x \in [x_0, x_1]$. The point ξ cannot exist. So $H_s'(x) > 0, \forall x \in [x_0, x_1]$. The result has been proven. □

13.4. INFORMATION DELAYS

We will study here the situation when the information is not obtained immediately, but after some delay. It is a particular case of partial information, however simpler. The conditional distribution of the inventory, given the information available can be replaced by some sufficient statistics, and thus the problem remains finite- dimensional. We will rely on the works **[6]**, **[7]**.

13.4.1. MODEL AND PRELIMINARIES. Let $\Omega, \mathcal{A}, P$ be a probability space. Time is discrete and there exists a process $D_t, t = 1, 2, \cdots$, of independent identically random variables. The CDF of a generic variable D is $F(x)$ and the probability density is $f(x)$. To each time t is associated a random variable θ_t representing the delay in receiving the information at time t. In particular the inventory at time $t-\theta_t$ will be known at time t provided this number is strictly positive. However $t-\theta_t$ is not necessarily the most recent time for which information is available at time t. The time of most recent information at time t is called the *vintage.* If we denote it by β_t then this process is defined by the evolution

$$\beta_t = \max(t-\theta_t, \beta_{t-1}),\ t \geq 2\ \beta_1 = 1. \tag{13.4.1}$$

The fact that $\beta_1 = 1$ means that we know the initial inventory. The inventory evolves as follows

$$y_{t+1} = y_t + v_t - D_t,\ y_1 = x. \tag{13.4.2}$$

We can introduce the filtration

$$\mathcal{F}^t = \sigma(D_1, \cdots D_{t-1}),$$

and in the situations studied before the order v_t was $\mathcal{F}^{t-1}$ measurable. This is no more the case. Call

$$z_t = y_{\beta_t}. \tag{13.4.3}$$

The information which is available at time t is the pair z_t, β_t and the previous values of this pair. We thus define the filtration of observations

$$\mathcal{Z}^t = \sigma((z_2, \beta_2), \cdots (z_t, \beta_t)),\ t \geq 2. \tag{13.4.4}$$

When $\theta_t = 0$, we have $\beta_t = t$ and $\mathcal{Z}^t = \mathcal{F}^t$. The order v_t is adapted to $\mathcal{Z}^t$. For $t = 1$ we have $\mathcal{Z}^1 = \{\Omega, \emptyset\}$ and $z_1 = y_1 = x$. We note moreover that

$$\beta_{t-1} \leq \beta_t \leq t. \tag{13.4.5}$$

To proceed with building the model, we have to make assumptions on the process of delays. First we shall assume that θ_t takes values in a finite state space $\mathcal{M} = \{0, 1, \cdots, M\}$. Moreover θ_t is a Markov chain with transition probability matrix $P = (p_{ij})$ with

$$p_{ij} = P(\theta_{t+1} = j | \theta_t = i).$$

We also assume that the two processes θ_t and D_t are *mutually independent.*

13.4.2. INVENTORY POSITION. We introduce the *inventory position*

$$x_t = \begin{cases} z_t, & \text{if } \beta_t = t \\ z_t + \sum_{j=\beta_t}^{t-1} v_j, & \text{if } \beta_t < t \end{cases} \tag{13.4.6}$$

We can prove the

Lemma 13.1. *The following relations hold*

$$x_{t+1} = x_t + v_t - 1\!\!1_{\beta_{t+1} > \beta_t} \sum_{j=\beta_t}^{\beta_{t+1}-1} D_j,\ \ x_1 = z_1 = y_1 = x \tag{13.4.7}$$

$$y_t = x_t - \sum_{j=\beta_t}^{t-1} D_j,\ \textit{if}\ \beta_t < t \tag{13.4.8}$$

The process x_t is observable.

PROOF. The fact that the inventory position is observable follows directly from its definition. The second relation follows from

$$\begin{aligned} x_t &= y_{\beta_t} + \sum_{j=\beta_t}^{t-1} v_j \\ y_t &= y_{\beta_t} + \sum_{j=\beta_t}^{t-1} v_j - \sum_{j=\beta_t}^{t-1} D_j \end{aligned}$$

To prove the first one, first note that

$$\begin{aligned} z_{t+1} - z_t &= y_{\beta_{t+1}} - y_{\beta_t} \\ &= \sum_{j=\beta_t}^{\beta_{t+1}-1} (v_j - D_j), \text{ if } \beta_t < \beta_{t+1} \end{aligned}$$

and (13.4.7) follows easily, by considering all the cases. □

Consider now the one period cost

$$c(x, v) = C(v) + l(x),$$

with

$$C(v) = K\mathbb{1}_{v>0} + cv, \;\; l(x) = hx^+ + px^-,$$

and a control

$$V = (v_1, \cdots, v_t, \cdots), \;\; v_t \text{ is } \mathcal{Z}^t \text{ measurable}.$$

The objective functional is

$$J(V) = E \sum_{t=1}^{\infty} \alpha^{t-1} c(y_t, v_t),$$

and from (13.4.8) we get

$$c(y_t, v_t) = c\left(x_t - \sum_{j=\beta_t}^{t} D_j, v_t \right) = C(v_t) + l\left(x_t - \sum_{j=\beta_t}^{t} D_j \right),$$

in which the sum vanishes when $\beta_t = t$. The process β_t is independent from the demand process. The demands being independent random variables, it is easy to check that $D_{\beta_t}, \cdots D_t$ are independent from $\mathcal{Z}^t$. Therefore if we introduce the notation

$$l^{(i)}(x) = El\left(x - \sum_{j=1}^{i} D_j \right), \; i \geq 1, \; l^{(0)}(x) = l(x),$$

then we can rewrite the objective functional as

$$J(V) = E \sum_{t=1}^{\infty} \alpha^{t-1} (C(v_t) + l^{(t-\beta_t)}(x_t)). \tag{13.4.9}$$

The important attribute of this writing is that the three processes x_t, v_t, β_t are $\mathcal{Z}^t$ measurable. However β_t is not a Markov chain, so it is not convenient to work with this process. We will introduce a new Markov chain, from which β_t is easily derived.

13.4.3. NEW MARKOV CHAIN. We introduce the *calendar time* which is the largest time before t when a delay has been observed. So

$$\tau_t = \max_{1 \leq j \leq t} (j : \beta_{j-1} < \beta_j),$$

therefore

$$\begin{aligned} \beta_{\tau_t} &= \tau_t - \theta_{\tau_t}; \\ \beta_{\tau_{t+1}} &= \cdots = \beta_t = \beta_{\tau_t}. \end{aligned}$$

We then set

$$\sigma_t = t - \tau_t, \ \ \eta_t = \theta_{\tau_t},$$

therefore the pair β_t, τ_t can be easily derived from the pair σ_t, η_t by the formulas

$$\beta_t = t - \sigma_t - \eta_t, \ \ \tau_t = t - \sigma_t. \tag{13.4.10}$$

Since $\beta_t \geq t - \theta_t$ we get $\sigma_t + \eta_t \leq \theta_t$, $\sigma_t \geq 0$, $\eta_t \geq 0$. Therefore the process σ_t, η_t takes values in $\mathcal{M} \times \mathcal{M}$. Moreover

$$\sigma_t + \eta_t \leq M, \ \ \sigma_t \leq t - 1.$$

The important result is the

Proposition 13.5. *The pair σ_t, η_t is a Markov chain on the state space $\mathcal{M} \times \mathcal{M}$ with the property $\sigma_t + \eta_t \leq M$. It is a process adapted to the filtration $\mathcal{Z}^t$.*

PROOF. We first define the transitions. Assume $\sigma_t = \sigma, \eta_t = \eta$ in which σ, η are two numbers satisfying $\sigma \geq 0, \eta \geq 0$, $\sigma + \eta \leq M$. Suppose $\sigma + \eta = M$. Then if $\beta_{t+1} = \beta_t$ we have

$$\tau_{t+1} = \tau_t = t - \sigma, \quad \sigma_{t+1} = t + 1 - \tau_{t+1} = \sigma + 1.$$

On the other hand $\eta_{t+1} = \eta$ and we get $\sigma_{t+1} + \eta_{t+1} = M + 1$ which is impossible. Therefore $\beta_{t+1} > \beta_t$ which implies $\tau_{t+1} = t + 1$ hence $\sigma_{t+1} = 0$. The variable η_{t+1} can take the values $0, \cdots M$. Next assume $\sigma + \eta < M$. We may have $\beta_{t+1} = \beta_t$ in which case $\sigma_{t+1} = \sigma + 1$ and $\eta_{t+1} = \eta$. Otherwise $\beta_{t+1} > \beta_t$ which implies $\sigma_{t+1} = 0$. But then

$$\eta_{t+1} = t + 1 - \beta_{t+1} < t + 1 - \beta_t = \sigma + \eta + 1.$$

Therefore η_{t+1} can take the values $0, \cdots \sigma + \eta$. So the transitions are summarized as follows:

If $\sigma + \eta < M$

$$\sigma, \eta \Rightarrow \left| \begin{array}{c} \sigma' = \sigma + 1, \eta' = \eta \\ \sigma' = 0, \eta' = k, \quad 0 \leq k \leq \sigma + \eta \end{array} \right. \tag{13.4.11}$$

If $\sigma + \eta = M$

$$\sigma, \eta \Rightarrow \sigma' = 0, \eta' = k, \quad 0 \leq k \leq M \tag{13.4.12}$$

We can now define the transition probability matrix. We first note $\theta_{t-\sigma} = \eta$, and

$$\begin{aligned} \beta_{\tau_t+1} &= t - \sigma - \eta = \max(\tau_t + 1 - \theta_{\tau_t+1}, \beta_{\tau_t}) \\ &= \max(t - \sigma + 1 - \theta_{t-\sigma+1}, t - \sigma - \eta), \end{aligned}$$

from which it follows $\theta_{t-\sigma+1} \geq \eta + 1$. Similarly

$$\theta_{t-\sigma+2} \geq \eta + 2, \cdots, \theta_t \geq \eta + \sigma.$$

Therefore if $\sigma + \eta < M$ we have

$$\begin{aligned} &P(\sigma_{t+1} = \sigma + 1, \eta_{t+1} = \eta | \sigma_t = \sigma, \eta_t = \eta) \\ &\quad = P(\beta_t \geq t + 1 - \theta_{t+1} | \sigma_t = \sigma, \eta_t = \eta) \\ &\quad = P(\theta_{t+1} \geq \sigma + \eta + 1 | \theta_{t-\sigma} = \eta, \theta_{t-\sigma+1} \geq \eta + 1, \cdots \theta_t \geq \eta + \sigma) \end{aligned}$$

Next for $0 \leq k \leq \sigma + \eta$

$$\begin{aligned} &P(\sigma_{t+1} = 0, \eta_{t+1} = k | \sigma_t = \sigma, \eta_t = \eta) \\ &\quad = P(\beta_{t+1} = t + 1 - \theta_{t+1}, \theta_{t+1} = k | \sigma_t = \sigma, \eta_t = \eta) \\ &\quad = P(\theta_{t+1} = k | \theta_{t-\sigma} = \eta, \theta_{t-\sigma+1} \geq \eta + 1, \cdots \theta_t \geq \eta + \sigma) \end{aligned}$$

We then define the function $\gamma(k; \sigma, \eta)$ with arguments $0 \leq k \leq M$, $\sigma + \eta \leq M$, $\sigma \geq 0$, $\eta \geq 0$, $\sigma \leq t - 1$

$$\gamma(k; \sigma, \eta) = P(\theta_{t+1} = k | \theta_{t-\sigma} = \eta, \theta_{t-\sigma+1} \geq \eta + 1, \cdots \theta_t \geq \eta + \sigma) \tag{13.4.13}$$

This function is uniquely defined by the Markov chain θ_t and its transition probability function. In fact it can be defined by the recursion

$$\gamma(k; \sigma, \eta) = \frac{\sum_{j=\sigma+\eta}^{M} p_{jk} \gamma(j; \sigma - 1, \eta)}{\sum_{j=\sigma+\eta}^{M} \gamma(j; \sigma - 1, \eta)}, \ \sigma \geq 1; \ \ \gamma(k; 0, \eta) = p_{\eta k} \tag{13.4.14}$$

The transition probability matrix of the Markov chain σ_t, η_t is given by:

If $\sigma + \eta < M$

$$\pi(\sigma, \eta; \sigma', \eta') = \begin{cases} \gamma(k; \sigma, \eta), & \text{if } \sigma' = 0, \eta' = k, 0 \leq k \leq \sigma + \eta \\ \sum_{k=\sigma+\eta+1}^{M} \gamma(k; \sigma, \eta), & \text{if } \sigma' = \sigma + 1, \eta' = \eta \\ 0, & \text{otherwise} \end{cases} \tag{13.4.15}$$

If $\sigma + \eta = M$

$$\pi(\sigma, \eta; \sigma', \eta') = \begin{cases} \gamma(k; \sigma, \eta), & \text{if } \sigma' = 0, \eta' = k, 0 \leq k \leq M \\ 0, & \text{otherwise} \end{cases} \tag{13.4.16}$$

The fact that σ_t, η_t is adapted to $\mathcal{Z}^t$ follows from the definition of the process. □

In fact, the Markov chain σ_t, η_t is not stationary from $t = 1$, because of the constraint $\sigma_t \leq t - 1$. It becomes stationary from $t \geq M + 1$. In the sequel we will discard this transitory phase and consider that we are in the stationary situation from the beginning. From (13.4.7) and (13.4.9) we can write

$$J(V) = E \sum_{t=1}^{\infty} \alpha^{t-1} (C(v_t) + l^{(\sigma_t + \eta_t)}(x_t)), \tag{13.4.17}$$

and

$$x_{t+1} = x_t + v_t - \mathbb{1}_{\sigma_{t+1} + \eta_{t+1} \leq \sigma_t + \eta_t} \sum_{j=\sigma_{t+1}+\eta_{t+1}}^{\sigma_t + \eta_t} D_{t,j} \tag{13.4.18}$$

where we have set $D_{t,j} = D_{t-j}$. There is the transitory period in which $\sigma_t + \eta_t < t$. It is over when $t \geq M + 1$. In (13.4.18) the random variables

$D_{t,j}$ are identically distributed, independent of $\mathcal{Z}^t$ and mutually independent. We discard the transitory period and consider in (13.4.17), (13.4.18) the Markov chain σ_t, η_t from $t = 1$ and we fix initial conditions by

$$x_1 = x, \ \sigma_1 = \sigma, \ \eta_1 = \eta, \ \sigma, \eta \geq 0, \ \sigma + \eta \leq M \tag{13.4.19}$$

We then call the functional $J(V)$ as $J_{x,\sigma,\eta}(V)$, to emphasize the initial conditions. We define

$$u(x,\sigma,\eta) = \inf_V J_{x,\sigma,\eta}(V). \tag{13.4.20}$$

If $M = 0$, then $\sigma_t = 0, \eta_t = 0$ and the problem reduces to the standard one with no delay in the information studied in Chapter 9, section 9.3.

13.4.4. DYNAMIC PROGRAMMING. It is easy to write the Bellman equation related to the value function (13.4.20). We will write it as follows

$$\begin{aligned}(13.4.21)\quad u(x;\sigma,\eta) = -cx + l^{(\sigma+\eta)}(x) + \inf_{y\geq x}\Bigg\{ &K1\!\!1_{y>x} + cy \\ &+\alpha\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)u^{(\sigma+\eta+1-k)}(y;0,k) + \sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)u(y;\sigma+1,\eta)\Bigg\}\end{aligned}$$

in which the last sum is dropped when $\sigma+\eta = M$. It is equivalent and convenient to write this equation as

$$\begin{aligned}(13.4.22)\quad u(x;\sigma,\eta) &= -cx + l^{(\sigma+\eta)}(x) \\ &+ \inf_{y\geq x}\left\{K1\!\!1_{y>x} + cy + \alpha E u^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\}\end{aligned}$$

where the expectation in the parenthesis refers to a random variable σ', η' which has a probability distribution $\pi(\sigma,\eta;\sigma',\eta')$, where σ,η are fixed.

Theorem 13.2. *We assume that*

$$c(1-\alpha) - \alpha p < 0, \tag{13.4.23}$$

then the solution of (13.5.9) *is unique in the space* B_1*, continuous, tends to* $+\infty$ *as* $|x| \to \infty$ *and is* $K-$*convex. The optimal feedback is defined by an* $s(\sigma,\eta)$, $S(\sigma,\eta)$ *policy.*

PROOF. The proof uses the methods of Chapter 9 and is not repeated. □

13.5. ERGODIC CONTROL WITH INFORMATION DELAYS

We want to study here the situation when $\alpha \to 1$. We follow the work [**7**].

13.5.1. ERGODIC PROPERTIES. We make the following assumption on the Markov chain θ_t

$$\varpi_0 = \min_{j\in\mathcal{M}} p_{j\,0} > 0. \tag{13.5.1}$$

This assumption guarantees that the Markov chain θ_t is ergodic, see Chapter 3, equation (3.6.8). We will deduce the

Proposition 13.6. *Assume* (13.5.1) *then the Markov chain* σ_t, η_t *is ergodic.*

PROOF. It is sufficient to show that

$$\min_{\sigma,\eta}\pi(\sigma,\eta;0.0) > 0. \tag{13.5.2}$$

But

$$\pi(\sigma,\eta;0.0) = \gamma(0;\sigma,\eta),$$

and from (13.4.14)

$$\gamma(0;\sigma,\eta) \geq \varpi_0 > 0,$$

hence the property (13.5.2). Let us define the corresponding invariant measure $\mu(\sigma,\eta)$. it is the solution of the following system,

$$\sum_{\substack{\sigma \geq 0 \quad \eta \geq 0 \\ \sigma+\eta \quad \leq M}} \mu(\sigma,\eta)\pi(\sigma,\eta;\sigma',\eta') = \mu(\sigma',\eta'), \tag{13.5.3}$$

using formulas (13.4.15), (13.4.16) we write

$$\sum_{\substack{\sigma \geq 0 \quad \eta \geq 0 \\ \eta' \leq \sigma+\eta \quad \leq M}} \mu(\sigma,\eta)\gamma(\eta';\sigma,\eta) = \mu(0,\eta'), \text{ if } \sigma'=0,\ 0 \leq \eta' \leq M \tag{13.5.4}$$

(13.5.5)

$$\mu(\sigma'-1,\eta') \sum_{k=\sigma'+\eta'}^{M} \gamma(k;\sigma'-1,\eta') = \mu(\sigma',\eta'), \text{ if } 1 \leq \sigma' \leq M,\ 0 \leq \eta' \leq M-\sigma'.$$

From (13.5.5) we get

$$\mu(\sigma,\eta) = \mu(0,\eta) \prod_{h=1,\cdots\sigma} \sum_{k=h+\eta}^{M} \gamma(k;h-1,\eta),\ 1 \leq \sigma \leq M,\ 0 \leq \eta \leq M-\sigma, \tag{13.5.6}$$

and $\mu(0,\eta)$ is obtained from (13.5.4) as solution of

$$\begin{aligned} \mu(0,\eta') = & \sum_{\eta=\eta'}^{M} \mu(0,\eta) \Bigg[\gamma(\eta';0,\eta) + \sum_{\sigma=1}^{M-\eta} \gamma(\eta';\sigma,\eta) \\ & \cdot \prod_{h=1,\cdots\sigma} \sum_{k=h+\eta}^{M} \gamma(k;h-1,\eta) \Bigg] + \sum_{\eta=0}^{\eta'-1} \mu(0,\eta) \sum_{\sigma=\eta'-\eta}^{M-\eta} \gamma(\eta';\sigma,\eta) \\ & \cdot \prod_{h=1,\cdots\sigma} \sum_{k=h+\eta}^{M} \gamma(k;h-1,\eta),\ 0 \leq \eta' \leq M \end{aligned} \tag{13.5.7}$$

in which the last term drops when $\eta' = 0$. This system of $M+1$ relations has in fact only M independent relations. We have the identity

$$\begin{aligned} & \sum_{\eta'=0}^{M} \mu(0,\eta') + \sum_{\eta'=0}^{M-1} \mu(0,\eta') \sum_{\sigma'=1}^{M-\eta'} \prod_{h=1,\cdots\sigma'} \sum_{k=h+\eta'}^{M} \gamma(k;h-1,\eta') \\ & = \sum_{\eta=0}^{M} \mu(0,\eta) + + \sum_{\eta=0}^{M-1} \mu(0,\eta) \sum_{\sigma'=1}^{M-\eta'} \prod_{h=1,\cdots\sigma} \sum_{k=h+\eta}^{M} \gamma(k;h-1,\eta) \end{aligned}$$

and this expression must be equal to 1. From ergodic theory this system has a unique solution such that $\mu(\sigma,\eta);\sigma,\eta \geq 0,\ \sigma+\eta \leq 1$ is a probability. □

13.5.2. ESTIMATES FOR THE CASE $K = 0$. We consider the Bellman equation

$$u_\alpha(x;\sigma,\eta) = -cx + l^{(\sigma+\eta)}(x) + \inf_{y\geq x}\left\{cy + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\} \tag{13.5.8}$$

where we have emphasized the dependence in α. Since $K = 0$, the optimal policy is a Base stock policy, defined by a base stock $S_\alpha(\sigma,\eta)$. The function of x,$u_\alpha(x;\sigma,\eta)$ is continuous, convex and tends to $+\infty$ as $|x| \to +\infty$. It is also described as follows

$$u_\alpha(x;\sigma,\eta) = -cx + l^{(\sigma+\eta)}(x) + cS_\alpha + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(S_\alpha;\sigma',\eta'),\ \forall x \leq S_\alpha(\sigma,\eta) \tag{13.5.9}$$

and

$$u_\alpha(x;\sigma,\eta) = l^{(\sigma+\eta)}(x) + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'),\ \forall x \geq S_\alpha(\sigma,\eta) \tag{13.5.10}$$

The function is differentiable as a consequence of convexity. Let us check that it is continuously differentiable at $S_\alpha(\sigma,\eta)$. Indeed

$$u'_\alpha(x;\sigma,\eta) = -c + (l')^{(\sigma+\eta)}(x),\ \forall x < S_\alpha(\sigma,\eta)$$

$$u'_\alpha(x;\sigma,\eta) = (l')^{(\sigma+\eta)}(x) + \alpha E(u'_\alpha)^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'),\ \forall x > S_\alpha(\sigma,\eta)$$

Since at S_α we have

$$-c = \alpha E(u'_\alpha)^{(\sigma+\eta+1-(\sigma'+\eta'))}(S_\alpha;\sigma',\eta'),$$

from the definition of the Base stock, the continuity of the derivative is obtained at S_α.

We now derive useful properties of the Base stock. We shall clarify first some abuse of notation. Applying the definition of $l^{(i)}(x)$ to the Heaviside function $\mathbb{1}_{x>0}$ we obtain the CDF of $D_1 + \cdots + D_i$, which has been denoted by $F^{(i)}(x)$ in Chapter 9, see equation (9.2.19). We will keep this notation for commodity, only when $F(x)$ represents the CDF of the variable D. We then state the

Proposition 13.7. *We have the estimate*

$$0 < S_\alpha(\sigma,\eta) < \bar{S}_\alpha(\sigma,\eta), \tag{13.5.11}$$

where $\bar{S}_\alpha(\sigma,\eta)$ is the solution of

$$F^{(\sigma+\eta+1)}(\bar{S}_\alpha) = \frac{\alpha(p+c)-c}{\alpha(h+p)}. \tag{13.5.12}$$

PROOF. We first prove the property

$$u'_\alpha(x;\sigma,\eta) \geq (l')^{(\sigma+\eta)}(x) - c. \tag{13.5.13}$$

We know it holds for $x \leq S_\alpha(\sigma,\eta)$. So we may assume $x > S_\alpha(\sigma,\eta)$. By the monotonicity of the derivative we have

$$(u'_\alpha)^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta') \geq (u'_\alpha)^{(\sigma+\eta+1-(\sigma'+\eta'))}(S_\alpha(\sigma,\eta);\sigma',\eta'),$$

therefore

$$\begin{aligned} u'_\alpha(x;\sigma,\eta) &\geq (l')^{(\sigma+\eta)}(x) + \alpha E(u'_\alpha)^{(\sigma+\eta+1-(\sigma'+\eta'))}(S_\alpha(\sigma,\eta);\sigma',\eta') \\ &= (l')^{(\sigma+\eta)}(x) - c, \end{aligned}$$

and thus (13.5.13) is obtained. Next from the relation

$$-c = \alpha E(u_{\alpha}^{'})^{(\sigma+\eta+1-(\sigma'+\eta'))}(S_\alpha(\sigma,\eta);\sigma',\eta'),$$

we obtain, using the inequality

$$\begin{aligned} -c &\geq -\alpha c + \alpha E((l')^{(\sigma'+\eta')})^{(\sigma+\eta+1-(\sigma'+\eta'))}(S_\alpha(\sigma,\eta)) \\ &= -\alpha c + \alpha (l')^{(\sigma+\eta+1)}(S_\alpha(\sigma,\eta)). \end{aligned}$$

But with our notation

$$(l')^{(\beta)}(x) = -p + (h+p)F^{(\beta)}(x),$$

and thus we obtain

$$0 \geq c(1-\alpha) - \alpha p + \alpha(h+p)F^{(\sigma+\eta+1)}(S_\alpha(\sigma,\eta)),$$

hence

$$F^{(\sigma+\eta+1)}(S_\alpha(\sigma,\eta)) \leq \frac{\alpha(p+c)-c}{\alpha(h+p)} = F^{(\sigma+\eta+1)}(\bar{S}_\alpha),$$

and the inequality on the right of (13.5.11) is obtained.

To prove the positivity consider

$$\underline{S_\alpha} = \min_{\sigma,\eta} S_\alpha(\sigma,\eta) = S_\alpha(\sigma_\alpha,\eta_\alpha)$$

The numbers $\sigma_\alpha, \eta_\alpha$ are well defined since we have only a finite number of possibilities. We can write

$$-c = \alpha E(u_{\alpha}^{'})^{(\sigma+\eta+1-(\sigma'+\eta'))}(\underline{S_\alpha};\sigma',\eta').$$

Since $\underline{S_\alpha} < S_\alpha(\sigma',\eta')$ we have

$$\begin{aligned} (u_{\alpha}^{'})^{(\sigma+\eta+1-(\sigma'+\eta'))}(\underline{S_\alpha};\sigma',\eta') &= -c + ((l')^{(\sigma'+\eta')})^{(\sigma+\eta+1-(\sigma'+\eta'))}(\underline{S_\alpha}) \\ &= -c + (l')^{(\sigma+\eta+1)}(\underline{S_\alpha}), \end{aligned}$$

hence

$$-c = -\alpha c + \alpha (l')^{(\sigma+\eta+1)}(\underline{S_\alpha}),$$

therefore also

$$F^{(\sigma_\alpha+\eta_\alpha+1)}(\underline{S_\alpha}) = \frac{\alpha(p+c)-c}{\alpha(h+p)},$$

which implies $\underline{S_\alpha} > 0$. This proves the left inequality (13.5.11). □

Using next

$$F^{(M+1)}(S_\alpha(\sigma,\eta)) \leq F^{(\sigma+\eta+1)}(S_\alpha(\sigma,\eta)) \leq \frac{\alpha(p+c)-c}{\alpha(h+p)} \leq \frac{p}{h+p},$$

we deduce that

(13.5.14) $$0 < S_\alpha(\sigma,\eta) \leq \bar{S},$$

with

$$F^{(M+1)}(\bar{S}) = \frac{p}{h+p}.$$

We next proceed with estimates on $u'_\alpha(x;\sigma,\eta)$. We are going to prove

Proposition 13.8. *The following estimate holds*

$$-c-p \leq u'_\alpha(x;\sigma,\eta) \leq h\left(1+\frac{1}{\varpi_0}\right)\frac{1}{\bar{F}(x)}. \tag{13.5.15}$$

So if $x<X$ then one can state that

$$-c-p \leq u'_\alpha(x;\sigma,\eta) \leq L_X,\ \ L_X = h\left(1+\frac{1}{\varpi_0}\right)\frac{1}{\bar{F}(X)}. \tag{13.5.16}$$

Proof. The estimate to the left follows directly from (13.5.13) and $(l')^{(\sigma+\eta)}(x) \geq -p$. We next assert that

$$\begin{aligned} u'_\alpha(x;\sigma,\eta) &\leq -c+h,\ \forall x \leq S_\alpha(\sigma,\eta) \\ &\leq h+\alpha E(u'_\alpha)^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'),\ \forall x > S_\alpha(\sigma,\eta) \end{aligned}$$

so also

$$(u'_\alpha)^+(x;\sigma,\eta) \leq h+\alpha E((u'_\alpha)^+)^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'),\ \forall x$$

This means, expliciting the mathematical expectation

$$(u'_\alpha)^+(x;\sigma,\eta) \leq h+(u'_\alpha)^+(x;\sigma+1,\eta)\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)$$
$$+\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)((u'_\alpha)^+)^{(\sigma+\eta+1-k)}(x;0,k).$$

However for $k \leq \sigma+\eta$ and from the monotonicity of $(u'_\alpha)^+$ we have

$$((u'_\alpha)^+)^{(\sigma+\eta+1-k)}(x;0,k) \leq E(u'_\alpha)^+(x-D;0,k),$$

therefore

$$\begin{aligned} (u'_\alpha)^+(x;\sigma,\eta) &\leq h+(u'_\alpha)^+(x;\sigma+1,\eta)\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta) \\ &\quad +\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)E(u'_\alpha)^+(x-D;0,k). \end{aligned}$$

Set

$$w_\alpha(x) = \sup_{\sigma,\eta}(u'_\alpha)^+(x;\sigma,\eta),$$

we deduce

$$\begin{aligned} (u'_\alpha)^+(x;\sigma,\eta) &\leq h+w_\alpha(x)\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)+Ew_\alpha(x-D)\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta) \\ &= h+w_\alpha(x)-\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)(w_\alpha(x)-Ew_\alpha(x-D)) \end{aligned}$$

We use

$$\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta) \geq \varpi_0,\ \ w_\alpha(x)-Ew_\alpha(x-D) \geq 0,$$

to deduce

$$(u'_\alpha)^+(x;\sigma,\eta) \leq h+w_\alpha(x)-\varpi_0(w_\alpha(x)-Ew_\alpha(x-D)),$$

from which we can obtain

$$w_\alpha(x) \leq h + w_\alpha(x) - \varpi_0(w_\alpha(x) - Ew_\alpha(x-D)).$$

We have then proved

$$w_\alpha(x) \leq Ew_\alpha(x-D) + \frac{h}{\varpi_0}.$$

Now

$$\begin{aligned} Ew_\alpha(x-D) &= E[w_\alpha(x-D)\mathbb{1}_{D\leq x}] + E[w_\alpha(x-D)\mathbb{1}_{D>x}] \\ &\leq w_\alpha(x)F(x)\mathbb{1}_{x>0} + E[w_\alpha(x-D)\mathbb{1}_{D>x}]. \end{aligned}$$

However

$$u'_\alpha(x;\sigma,\eta) \leq h,\ \forall x \leq 0 \Rightarrow w_\alpha(x) \leq h,\ \forall x \leq 0$$

from which it follows that

$$Ew_\alpha(x-D) \leq w_\alpha(x^+)F(x^+) + h,$$

which implies

$$w_\alpha(x)\bar{F}(x) \leq h\left(1 + \frac{1}{\varpi_0}\right),$$

and the proof has been completed. □

13.5.3. LIMIT BEHAVIOR FOR THE CASE $K = 0$. We can now state the limit of (13.5.39).

Theorem 13.3. *Assume* (13.5.1) *and*

$$\bar{F}(x) > 0, \forall x < \infty, \quad p > c. \tag{13.5.17}$$

There exists a sequence ρ_α such that if $\alpha \to 1$, for a subsequence

$$u_\alpha(x;\sigma,\eta) - \frac{\rho_\alpha}{1-\alpha} \to u(x;\sigma,\eta), \tag{13.5.18}$$

uniformly on any compact of R. We have also $\rho_\alpha \to \rho$. The function $u(x;\sigma,\eta)$ is locally Lipschitz continuous and satisfies

$$|u'(x;\sigma,\eta)| \leq \frac{C}{\bar{F}(x)}.$$

The pair $u(x;\sigma,\eta), \rho$ satisfies

$$\begin{aligned} u(x;\sigma,\eta) + \rho = &-cx + l^{(\sigma+\eta)}(x) \\ &+ \inf_{y\geq x}\left\{cy + \sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)u^{(\sigma+\eta+1-k)}(y;0,k)\right. \\ &\left. + \sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)u(y;\sigma+1,\eta)\right\}. \end{aligned} \tag{13.5.19}$$

PROOF. Consider $u_\alpha(0;\sigma,\eta)$. Since $S_\alpha(\sigma,\eta)>0$, we can write, noting that $l^{(\sigma+\eta)}(0)=p\bar{D}(\sigma+\eta)$

$$\begin{aligned}u_\alpha(0;\sigma,\eta) &= p\bar{D}(\sigma+\eta)+cS_\alpha(\sigma,\eta)\\ &+\alpha\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)u_\alpha(S_\alpha(\sigma,\eta);\sigma+1,\eta)\\ &+\alpha\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)Eu_\alpha(S_\alpha(\sigma,\eta)-D^{(\sigma+\eta+1-k};0,k).\end{aligned}$$

Set

$$\begin{aligned}G_\alpha(\sigma,\eta) &= u_\alpha(0;\sigma,\eta)-\alpha\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)u_\alpha(0;\sigma+1,\eta)\\ &-\alpha\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)u_\alpha(0;0,k).\end{aligned}$$

Using the value of $u_\alpha(0;\sigma,\eta)$ we can state

$$\begin{aligned}G_\alpha(\sigma,\eta) &= p\bar{D}(\sigma+\eta)+cS_\alpha(\sigma,\eta)\\ &+\alpha\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)(u_\alpha(S_\alpha(\sigma,\eta);\sigma+1,\eta)-u_\alpha(0;\sigma+1,\eta))\\ &+\alpha\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)E(u_\alpha(S_\alpha(\sigma,\eta)-D^{(\sigma+\eta+1-k};0,k)-u_\alpha(0;0,k)).\end{aligned}$$

Using both estimates (13.5.14), (13.5.15) we deduce easily that $|G_\alpha(\sigma,\eta)|\le C$. The definition of $G_\alpha(\sigma,\eta)$ can also be written as

$$u_\alpha(0;\sigma,\eta)-\alpha\sum_{\sigma',\eta'}\pi(\sigma,\eta;\sigma',\eta')u_\alpha(0;\sigma',\eta')=G_\alpha(\sigma,\eta) \tag{13.5.20}$$

The chain $\pi(\sigma,\eta;\sigma',\eta')$ is ergodic and its invariant measure is $\mu(\sigma,\eta)$. Define

$$\rho_\alpha=\sum_{\sigma,\eta}\mu(\sigma,\eta)G_\alpha(\sigma,\eta).$$

We also have since $\mu(\sigma,\eta)$ is invariant

$$\sum_{\sigma,\eta}\mu(\sigma,\eta)u_\alpha(0;\sigma,\eta)=\frac{\rho_\alpha}{1-\alpha}.$$

From ergodic theory we can also assert that

$$\left|u_\alpha(0;\sigma,\eta)-\frac{\rho_\alpha}{1-\alpha}\right|\le C, \tag{13.5.21}$$

where the constant is independent of α,σ,η. Define next

$$\bar{u}_\alpha(x;\sigma,\eta)=u_\alpha(x;\sigma,\eta)-\frac{\rho_\alpha}{1-\alpha}.$$

Combining (13.5.21) with (13.5.15) we can state the following estimates

$$|\bar{u}_\alpha(x;\sigma,\eta)|\le C+C\frac{|x|}{\bar{F}(x)}; \tag{13.5.22}$$

(13.5.23) $$|(\bar{u}_\alpha)'(x;\sigma,\eta)| \leq \frac{C}{\bar{F}(x)}.$$

More generally

(13.5.24) $$|(\bar{u}_\alpha)^{(\beta)}(x;\sigma,\eta)| \leq C + C\frac{|x| + \beta\bar{D}}{\bar{F}(x)};$$

(13.5.25) $$|((\bar{u}_\alpha)^{(\beta)})'(x;\sigma,\eta)| \leq \frac{C}{\bar{F}(x)}.$$

From Ascoli-Arzelà Theorem, we can find a subsequence such that

$$\begin{aligned}
\rho_\alpha \quad &\to \quad \rho, \sup_{|x|\leq X} |\bar{u}_\alpha(x;\sigma,\eta) - u(x;\sigma,\eta)| \to 0 \\
&\quad \sup_{|x|\leq X} |(\bar{u}_\alpha)^{(\beta)}(x;\sigma,\eta) - u^{(\beta)}(x;\sigma,\eta)| \to 0 \\
&\forall \quad \sigma,\eta,\beta,X \text{ as } \alpha \to 1
\end{aligned}$$

From the estimate (13.5.25) the limit satisfies

$$|u'(x;\sigma,\eta)| \leq \frac{C}{\bar{F}(x)}.$$

Going back to (13.5.27) and recalling that the infimum in y is attained at $\max(x, S_\alpha(\sigma,\eta))$ we can restrict y to $y \leq L_X$ provided $x < X$. So we can write using $\bar{u}_\alpha(x;\sigma,\eta)$

$$\begin{aligned}
\bar{u}_\alpha(x;\sigma,\eta) + \rho_\alpha &= -cx + l^{(\sigma+\eta)}(x) \\
&+ \inf_{L_X \geq y \geq x} \left\{cy + \alpha E\bar{u}_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\},
\end{aligned}$$

and thus using the uniform convergence on compact sets of R we obtain

$$\begin{aligned}
u(x;\sigma,\eta) + \rho &= -cx + l^{(\sigma+\eta)}(x) \\
&+ \inf_{L_X \geq y \geq x} \left\{cy + Eu^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\} \\
&\geq -cx + l^{(\sigma+\eta)}(x) + \inf_{y\geq x} \left\{cy + Eu^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\}.
\end{aligned}$$

On the other hand for any $y > x$ we have also

$$u(x;\sigma,\eta) + \rho \leq -cx + l^{(\sigma+\eta)}(x)cy + Eu^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta'),$$

and thus (13.5.39) is obtained. The proof has been completed. □

Consider the cost functional

$$J^\alpha_{x,\sigma,\eta}(V) = E\sum_{t=1}^{\infty} \alpha^{t-1}(cv_t + l^{(\sigma_t+\eta_t)}(x_t)),$$

then we shall have

(13.5.26) $$\rho = \inf_V \lim_{\alpha\to 1}(1-\alpha)J^\alpha_{x,\sigma,\eta}(V),$$

considering only control policies V such that the limit exists. Then the infimum is attained at $\hat{V}$ characterized by a Base stock

$$S(\sigma,\eta) = \lim_{\alpha\to 1} S_\alpha(\sigma,\eta).$$

13.5.4. THE CASE $K > 0$. The equation is written as follows

$$u_\alpha(x;\sigma,\eta) = -cx + l^{(\sigma+\eta)}(x) + \inf_{y\geq x}\left\{K\mathbb{1}_{y>x} + cy + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\}. \tag{13.5.27}$$

The solution is governed by an $s_\alpha(\sigma,\eta),\ S_\alpha(\sigma,\eta)$ policy. Hence

$$x \leq s_\alpha(\sigma,\eta) \Rightarrow u_\alpha(x;\sigma,\eta) = -cx + l^{(\sigma+\eta)}(x) + K + C_{\alpha,\sigma,\eta},$$

and

$$x \geq s_\alpha(\sigma,\eta) \Rightarrow u_\alpha(x;\sigma,\eta) = l^{(\sigma+\eta)}(x) + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta').$$

We also have the continuity condition

$$C_{\alpha,\sigma,\eta} = \inf_{y>s_\alpha(\sigma,\eta)}\left\{cy + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta')\right\}.$$

The function $u_\alpha(x;\sigma,\eta)$ is continuous but not continuously differentiable. The only point of discontinuity of the derivative is $s_\alpha(\sigma,\eta)$. However we can consider the discontinuous function $u_\alpha(x;\sigma,\eta)$ and write

$$u'_\alpha(x;\sigma,\eta) = -c + l'^{(\sigma+\eta)}(x) + \chi_\alpha(x;\sigma,\eta).$$

After some calculations we can write the equations

$$\chi_\alpha(x;\sigma,\eta) = \begin{cases}\mu_\alpha^{(\sigma+\eta)}(x) + \alpha E\chi_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'), & x \geq s_\alpha(\sigma,\eta)\\ 0, & x \leq s_\alpha(\sigma,\eta)\end{cases} \tag{13.5.28}$$

and

$$\mu_\alpha(x) = c(1-\alpha) + \alpha h - \alpha(h+p)\bar{F}(x). \tag{13.5.29}$$

The continuity condition is then expressed as follows

$$0 = K + \inf_{y>s_\alpha(\sigma,\eta)}\int_{s_\alpha(\sigma,\eta)}^{y}\chi_\alpha(\xi;\sigma,\eta)d\xi \tag{13.5.30}$$

If we define

$$G_\alpha(x;\sigma,\eta) = cx + \alpha E u_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'), \tag{13.5.31}$$

then we can write

$$u_\alpha(x;\sigma,\eta) = -cx + l^{(\sigma+\eta)}(x) + G_\alpha(\max(x,s_\alpha(\sigma,\eta));\sigma,\eta), \tag{13.5.32}$$

and thus

$$\chi_\alpha(x;\sigma,\eta) = G'_\alpha(x;\sigma,\eta)\mathbb{1}_{x>s_\alpha(\sigma,\eta)}. \tag{13.5.33}$$

We state now an important property

Lemma 13.2. *Assume $\alpha p > c$. We have the properties*

$$\begin{aligned} u'_\alpha(x;\sigma,\eta) &\leq -(c+p),\ \chi'_\alpha(x;\sigma,\eta) \leq 0 \quad \forall x<0\\ G'_\alpha(x;\sigma,\eta) &\leq c(1-\alpha) - \alpha p \quad \forall x<0\end{aligned}$$

PROOF. Note that if we know that $G'_\alpha(x;\sigma,\eta) < 0$, for $x < 0$, then the result follows immediately from formulas (13.5.31), (13.5.32) after taking the derivative in x. To complete we consider an iterative process for equations (13.5.31), (13.5.32), starting with $u_{0\alpha}(x;\sigma,\eta) = 0$. We then have

$$\begin{aligned} G_{0\alpha}(x;\sigma,\eta) &= cx,\ s_{0\alpha}(\sigma,\eta) = -\infty; \\ u_{1\alpha}(x;\sigma,\eta) &= l^{(\sigma+\eta)}(x), \end{aligned}$$

therefore for $x < 0$,

$$G_{1\alpha}(x;\sigma,\eta) = cx - \alpha px + p\bar{D}(\sigma+\eta+1),$$

which implies $G'_{1\alpha}(x;\sigma,\eta) < 0$, for $x < 0$. The iteration preserves the properties. This completes the proof. □

We then proceed with estimates.

Proposition 13.9. *We have the estimates*

$$\begin{aligned} s^-_\alpha(\sigma,\eta) &\le \frac{K}{\alpha p - c(1-\alpha)} = m_\alpha, \\ 0 \le S_\alpha(\sigma,\eta) &\le \frac{\alpha p m_\alpha + \alpha(p+h)\bar{D}(\sigma+\eta+1)}{c(1-\alpha)+\alpha h}. \end{aligned}$$

PROOF. We begin with the first inequality. it is sufficient to assume $s_\alpha(\sigma,\eta) < 0$. Note that from (13.5.33) and Lemma 13.2 we have

$$\chi_\alpha(x;\sigma,\eta) \le c(1-\alpha) - \alpha p,\ \forall s_\alpha(\sigma,\eta) < x < 0$$

Next from (13.5.30), since $s_\alpha(\sigma,\eta) < 0$,

$$0 \le K + \int_{s_\alpha(\sigma,\eta)}^0 \chi_\alpha(\xi;\sigma,\eta)d\xi,$$

and the the first estimate follows immediately. Let us proceed with the second inequality.

It is useful to recall the equivalence

$$Eu_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta') = E[u_\alpha(x - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta')|\sigma,\eta]$$

We also recall the optimal feedback

$$\hat{v}_\alpha(x;\sigma,\eta) = \begin{cases} S_\alpha(\sigma,\eta) - x, & \text{if } x \le s_\alpha(\sigma,\eta) \\ 0, & \text{if } x > s_\alpha(\sigma,\eta) \end{cases}$$

The Bellman equation is then

$$\begin{aligned} u_\alpha(x;\sigma,\eta) = l^{(\sigma+\eta)}(x) + K 1\!\!1_{\hat{v}_\alpha(x;\sigma,\eta)>0} + c\hat{v}_\alpha(x;\sigma,\eta) \\ +E[u_\alpha(x + \hat{v}_\alpha(x;\sigma,\eta) - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta')|\sigma,\eta] \end{aligned}$$

Let us denote

$$\hat{y}_2 = S_\alpha(\sigma,\eta) - D^{(\sigma+\eta+1-(\sigma'+\eta'))},\ \ \hat{v}_2 = \hat{v}_\alpha(x;\sigma',\eta').$$

In $\hat{y}_2, \hat{v}_2$ the randomness comes from the demands and σ',η', whereas σ,η is known. Since $s_\alpha(\sigma,\eta) > -m_\alpha$ we can write

$$\begin{aligned} u_\alpha(-m_\alpha;\sigma,\eta) = (p+c)m_\alpha + p(\sigma+\eta)\bar{D} + K \\ +cS_\alpha(\sigma,\eta) + \alpha E[u_\alpha(\hat{y}_2;\sigma',\eta')|\sigma,\eta]. \end{aligned}$$

On the other hand

$$u_\alpha(\hat{y}_2;\sigma',\eta') = l^{(\sigma'+\eta')}(\hat{y}_2) + K\mathbb{1}_{\hat{v}_2>0} + c\hat{v}_2$$
$$+\alpha E[u_\alpha(\hat{y}_2+\hat{v}_2 - D^{(\sigma'+\eta'+1-(\sigma"+\eta"))};\sigma",\eta")|\sigma',\eta'].$$

Combining these two relations and using rules of expected values we obtain

$$u_\alpha(-m_\alpha;\sigma,\eta) = (p+c)m_\alpha + p(\sigma+\eta)\bar{D} + K + cS_\alpha(\sigma,\eta)$$
$$+\alpha E\Big\{l^{(\sigma'+\eta')}(\hat{y}_2) + K\mathbb{1}_{\hat{v}_2>0} + c\hat{v}_2$$
$$+\alpha E(u_\alpha(\hat{y}_2+\hat{v}_2 - D^{(\sigma'+\eta'+1-(\sigma"+\eta"))};\sigma",\eta")|\sigma,\eta\Big\}. \tag{13.5.34}$$

On the other hand we can write the following two inequalities, from Bellman equation

$$u_\alpha(-m_\alpha;\sigma,\eta) \leq pm_\alpha + p(\sigma+\eta)\bar{D}$$
$$+\alpha E\left[u_\alpha(-m_\alpha - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta')|\sigma,\eta\right]$$

$$u_\alpha(-m_\alpha - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta') \leq l^{(\sigma'+\eta')}(-m_\alpha - D^{(\sigma+\eta+1-(\sigma'+\eta'))})$$
$$+K + c(\hat{v}_2+\hat{y}_2+m_\alpha + D^{(\sigma+\eta+1-(\sigma'+\eta'))})$$
$$+\alpha E[u_\alpha(\hat{y}_2+\hat{v}_2 - D^{(\sigma'+\eta'+1-(\sigma"+\eta"))};\sigma",\eta")|\sigma',\eta'],$$

hence conditioning

$$E[u_\alpha(-m_\alpha - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta')|\sigma,\eta] \leq K + p\bar{D}(\sigma+\eta+1) + (p+c)m_\alpha$$
$$+cE[\hat{v}_2+\hat{y}_2|\sigma,\eta] + c\bar{D}(\sigma+\eta+1-E(\sigma'+\eta'|\sigma,\eta))$$
$$+\alpha E[u_\alpha(\hat{y}_2+\hat{v}_2 - D^{(\sigma'+\eta'+1-(\sigma"+\eta"))};\sigma",\eta")|\sigma,\eta]$$

Collecting results we obtain the inequality

$$u_\alpha(-m_\alpha;\sigma,\eta) \leq p(m_\alpha + (\sigma+\eta)\bar{D}) + \alpha K$$
$$+\alpha p\bar{D}(\sigma+\eta+1) + \alpha(p+c)m_\alpha + \alpha cE[\hat{v}_2+\hat{y}_2|\sigma,\eta]$$
$$+\alpha c\bar{D}(\sigma+\eta+1-E(\sigma'+\eta'|\sigma,\eta))$$
$$+\alpha^2 E[u_\alpha(\hat{y}_2+\hat{v}_2 - D^{(\sigma'+\eta'+1-(\sigma"+\eta"))};\sigma",\eta")|\sigma,\eta]. \tag{13.5.35}$$

Comparing equation (13.5.34) and inequality (13.5.35) and canceling terms we get

$$c(1-\alpha)S_\alpha(\sigma,\eta) + K(1-\alpha) + \alpha KE[\mathbb{1}_{\hat{v}_2>0}|\sigma,\eta] + c(1-\alpha)E[\hat{v}_2|\sigma,\eta]$$
$$+\alpha hE(S_\alpha(\sigma,\eta) - D^{(\sigma+\eta+1)})^+ + \alpha pE(S_\alpha(\sigma,\eta) - D^{(\sigma+\eta+1)})^-$$
$$\leq (\alpha p - c(1-\alpha))m_\alpha + \alpha p\bar{D}(\sigma+\eta+1),$$

hence also

$$(c(1-\alpha)+\alpha h)S_\alpha(\sigma,\eta) \leq (\alpha p - c(1-\alpha))m_\alpha + \alpha(p+\bar{h})D(\sigma+\eta+1),$$

and the second inequality is proven. □

We can deduce estimates which do not depend on α. Since $\alpha \to 1$ we may assume $\alpha > \frac{1}{2}$. We state

Corollary 13.1. *For* $\alpha > \frac{1}{2}$ *we have*

$$|s_\alpha(\sigma,\eta)|,\ S_\alpha(\sigma,\eta) \le m_0$$

with

$$m_0 = \frac{\frac{2pK}{p-c} + (p+h)\bar{D}(M+1)}{\min(c,h)}$$

PROOF. It is an immediate consequence of Proposition 13.9. □

We now prove estimates on $u'_\alpha(x;\sigma,\eta)$.

Lemma 13.3. *We shall prove the estimate*

$$\sup_{\substack{\xi < x \\ \sigma,\eta}} |u'_\alpha(x;\sigma,\eta)| \le \frac{C}{\bar{F}((x+m_0)^+)}. \tag{13.5.36}$$

PROOF. We recall that

$$\begin{aligned}\chi_\alpha(x;\sigma,\eta) &= \mu_\alpha^{(\sigma+\eta)}(x) + \alpha E\chi_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta'),\ \forall x > s_\alpha(\sigma,\eta)\\ &= 0, \quad \forall x \le s_\alpha(\sigma,\eta)\end{aligned}$$

with

$$\mu_\alpha(\sigma,\eta) = c(1-\alpha) + \alpha h - \alpha(h+p)\bar{F}(x),$$

and

$$|s_\alpha(\sigma,\eta)|,\ S_\alpha(\sigma,\eta) \le m_0$$

We use

$$\begin{aligned}&E\chi_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(x;\sigma',\eta')\\ &\quad = E[\chi_\alpha(x - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta')\mathbb{1}_{x-D^{(\sigma+\eta+1-(\sigma'+\eta'))}>s_\alpha(\sigma',\eta')}].\end{aligned}$$

Since

$$\mathbb{1}_{x-D^{(\sigma+\eta+1-(\sigma'+\eta'))}>s_\alpha(\sigma',\eta')} \le \mathbb{1}_{x-D^{(\sigma+\eta+1-(\sigma'+\eta'))}>-m_0},$$

we deduce

$$\begin{aligned}&|\chi_\alpha(x;\sigma,\eta)|\mathbb{1}_{x>-m_0} \le |\mu_\alpha^{(\sigma+\eta)}(x)|\\ &\quad +\alpha E[|\chi_\alpha(x - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta')|\mathbb{1}_{x-D^{(\sigma+\eta+1-(\sigma'+\eta'))}>-m_0}],\end{aligned}$$

using $|\mu_\alpha^{(\sigma+\eta)}(x)| \le \max(h,p)$ and expanding the expectation we get

$$\begin{aligned}&|\chi_\alpha(x;\sigma,\eta)|\mathbb{1}_{x>-m_0} \le \max(h,p) +\\ &+\alpha|\chi_\alpha(x;\sigma+1,\eta)|\mathbb{1}_{x>-m_0}\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta)\\ &+\alpha\sum_{k=0}^{\sigma+\eta}E|\chi_\alpha(x - D^{(\sigma+\eta+1-k)};0,k)|\mathbb{1}_{x-D^{(\sigma+\eta+1-k)}>-m_0}\gamma(k;\sigma,\eta).\end{aligned}$$

Let us denote

$$w_\alpha(x) = \sup_{\substack{\xi < x \\ \sigma,\eta}} |\chi_\alpha(\xi;\sigma,\eta)|\mathbb{1}_{x>-m_0}.$$

We get from the previous inequality

$$|\chi_\alpha(x;\sigma,\eta)|\mathbb{1}_{x>-m_0}$$
$$\leq \max(h,p) + w_\alpha(x)\sum_{k=\sigma+\eta+1}^{M}\gamma(k;\sigma,\eta) + Ew_\alpha(x-D)\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta)$$
$$= \max(h,p) + w_\alpha(x) - (w_\alpha(x) - Ew_\alpha(x-D))\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta),$$

and using $w_\alpha(x) - Ew_\alpha(x-D) \geq 0$, $\sum_{k=0}^{\sigma+\eta}\gamma(k;\sigma,\eta) \geq \varpi_0$, we obtain

$$|\chi_\alpha(x;\sigma,\eta)|\mathbb{1}_{x>-m_0} \leq \max(h,p) + w_\alpha(x) - (w_\alpha(x) - Ew_\alpha(x-D))\varpi_0$$

Let x_0 be fixed. For $x < x_0$ we may also write

$$|\chi_\alpha(x;\sigma,\eta)|\mathbb{1}_{x>-m_0} \leq \max(h,p) + w_\alpha(x_0) - (w_\alpha(x_0) - Ew_\alpha(x_0-D))\varpi_0,$$

and thus

$$w_\alpha(x_0) \leq \max(h,p) + w_\alpha(x_0) - (w_\alpha(x_0) - Ew_\alpha(x_0-D))\varpi_0,$$

which implies

$$w_\alpha(x) - Ew_\alpha(x-D) \leq \frac{\max(h,p)}{\varpi_0}$$
$$w_\alpha(x) - Ew_\alpha(x-D)\mathbb{1}_{x-D>-m_0} \leq \frac{\max(h,p)}{\varpi_0}.$$

Finally

$$w_\alpha(x) \leq \frac{\max(h,p)}{\varpi_0}\frac{1}{\bar{F}((x+m_0)^+)}.$$

From this estimate the results follows immediately. □

Define then

$$\Gamma_\alpha(\sigma,\eta) = u_\alpha(-m_0;\sigma,\eta).$$

Since $-m_0 < s_\alpha(\sigma,\eta)$ we may write

$$u_\alpha(-m_0;\sigma,\eta) = (p+c)m_0 + p\bar{D}(\sigma+\eta) + cs_\alpha(\sigma,\eta)$$
$$+\alpha Eu_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(s_\alpha(\sigma,\eta);\sigma',\eta'),$$

and thus we obtain

$$\Gamma_\alpha(\sigma,\eta) = \alpha E\Gamma_\alpha(\sigma',\eta') + \Psi_\alpha(\sigma,\eta),$$

with

$$\Psi_\alpha(\sigma,\eta) = (p+c)m_0 + p\bar{D}(\sigma+\eta) + cs_\alpha(\sigma,\eta)$$
$$+\alpha E[u_\alpha(s_\alpha(\sigma,\eta) - D^{(\sigma+\eta+1-(\sigma'+\eta'))};\sigma',\eta') - u_\alpha(-m_0;\sigma,\eta)].$$

From the estimates of Proposition 13.9 and Lemma 13.3 we see immediately that

$$|\Psi_\alpha(\sigma,\eta)| \leq C,\ \forall\alpha,\sigma,\eta.$$

Using the invariant measure $\mu(\sigma,\eta)$ we define

$$\rho_\alpha = \sum_{\sigma\,\eta}\Psi_\alpha(\sigma,\eta)\mu(\sigma,\eta),$$

and thus from ergodic theory

$$\left|\Gamma_\alpha(\sigma,\eta) - \frac{\rho_\alpha}{1-\alpha}\right|. \tag{13.5.37}$$

We can then state the

Theorem 13.4. *We make the assumptions of Theorem 13.3.*

There exists a sequence ρ_α such that if $\alpha \to 1$, for a subsequence

$$u_\alpha(x;\sigma,\eta) - \frac{\rho_\alpha}{1-\alpha} \to u(x;\sigma,\eta), \tag{13.5.38}$$

uniformly on any compact of R. We have also $\rho_\alpha \to \rho$. The function $u(x;\sigma,\eta)$ is locally Lipschitz continuous and satisfies

$$|u'(x;\sigma,\eta)| \leq \frac{C}{\bar{F}((x+m_0)^+)}$$

The pair $u(x;\sigma,\eta), \rho$ satisfies

$$\begin{aligned} u(x;\sigma,\eta) + \rho &= -cx + l^{(\sigma+\eta)}(x) \\ &+ \inf_{y\geq x}\left\{ K \mathbb{1}_{y>x} + cy + \sum_{k=0}^{\sigma+\eta} \gamma(k;\sigma,\eta) u^{(\sigma+\eta+1-k)}(y;0,k) \right. \\ &\left. + \sum_{k=\sigma+\eta+1}^{M} \gamma(k;\sigma,\eta) u(y;\sigma+1,\eta) \right\} \end{aligned} \tag{13.5.39}$$

Proof. We define again

$$\bar{u}_\alpha(x;\sigma,\eta) = u_\alpha(x;\sigma,\eta) - \frac{\rho_\alpha}{1-\alpha},$$

then we get the equation

$$\begin{aligned} \bar{u}_\alpha(x;\sigma,\eta) + \rho_\alpha &= -cx + l^{(\sigma+\eta)}(x) \\ &+ \inf_{y\geq x}\left\{ K\mathbb{1}_{y>x} + cy + \alpha E \bar{u}_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta') \right\} \end{aligned}$$

Since the infimum is attained at $\max(x, S_\alpha(\sigma,\eta))$, we can consider that $y < L_X$ if $x < X$. Therefore we have also

$$\begin{aligned} \bar{u}_\alpha(x;\sigma,\eta) + \rho_\alpha &= -cx + l^{(\sigma+\eta)}(x) \\ &+ \inf_{L_X\geq y\geq x}\left\{ K\mathbb{1}_{y>x} + cy + \alpha E \bar{u}_\alpha^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta') \right\}. \end{aligned} \tag{13.5.40}$$

Assuming $|x| < X$, using the estimates of Proposition 13.9 and Lemma 13.3, and applying the Ascoli-Arzelà theorem we can extract a subsequence such that

$$\rho_\alpha \to \rho$$

$$\sup_{|x|<X} |\bar{u}_\alpha(x;\sigma,\eta) - u(x;\sigma,\eta)| \to 0$$

and we can pass to the limit in (13.5.40) to obtain

$$\begin{aligned} &u(x;\sigma,\eta) + \rho \\ &\quad = -cx + l^{(\sigma+\eta)}(x) + \inf_{L_X\geq y\geq x}\left\{ K\mathbb{1}_{y>x} + cy + \alpha E u^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta') \right\} \\ &\quad \geq -cx + l^{(\sigma+\eta)}(x) + \inf_{y\geq x}\left\{ K\mathbb{1}_{y>x} + cy + \alpha E u^{(\sigma+\eta+1-(\sigma'+\eta'))}(y;\sigma',\eta') \right\}. \end{aligned}$$

But the opposite inequality can be checked easily. The result follows. □

Consider the cost functional

$$J^{\alpha}_{x,\sigma,\eta}(V) = E\sum_{t=1}^{\infty}\alpha^{t-1}(C(v_t) + l^{(\sigma_t+\eta_t)}(x_t)),$$

then we shall have

$$\rho = \inf_{V}\lim_{\alpha\to 1}(1-\alpha)J^{\alpha}_{x,\sigma,\eta}(V) \tag{13.5.41}$$

considering only control policies V such that the limit exists. Then the infimum is attained at $\hat{V}$ characterized by an $s(\sigma,\eta),\ S(\sigma,\eta)$ policy with

$$s(\sigma,\eta) = \lim_{\alpha\to 1} s_{\alpha}(\sigma,\eta), \ \ S(\sigma,\eta) = \lim_{\alpha\to 1} S_{\alpha}(\sigma,\eta)$$

CHAPTER 14

CONTINUOUS TIME INVENTORY CONTROL

14.1. DETERMINISTIC MODEL

14.1.1. IMPULSE CONTROL IN CONTINUOUS TIME. We describe a continuous time model. An impulse control is a succession of times

$$\theta_1 < \theta_2 < \cdots \theta_n \cdots ,$$

at which we make decisions

$$v_1 < v_2 < \cdots v_n \cdots .$$

The times θ_n will be the times when we decide to replenish an inventory, and the v_n will be the levels of replenishment. In this deterministic framework, these numbers are deterministic. We represent the sequence of times and levels by V, and call V an impulse control.

Let $y(t) = y_x(t; V)$ be the state of the inventory at time t, with initial state x at time 0, and subject to an impulse control V.

We have

$$y(t) = x - \lambda t + M(t),$$

with

$$M(t) = M(t; V) = \sum_{\{n|\theta_n < t\}} v_n.$$

The meaning of these relations is clear: $x > 0$ is the initial inventory, λt is the demand up to time t and $M(t)$ represents the total replenishment up to time t. Since shortage is not allowed, we impose a state constraint $y(t) \geq 0$. We define a cost function as follows

$$J_x(V) = \sum_{n=0}^{\infty}(K + cv_n) \exp -\alpha\theta_n + \int_0^{\infty} hy(t) \exp -\alpha t dt.$$

As usual, we define the value function by

$$u(x) = \inf_V J_x(V).$$

A feedback is a decision rule which defines at any time t the decision to be taken as a function of the state of the system at that time. In our case, at any time t, looking at the value of the inventory $y(t)$ we decide to put an order and we decide the quantity ordered.

14.1.2. s, S POLICY. An s, S policy is a pair of numbers with

$$s \leq S,$$

such that a feedback is obtained from these two numbers as follows:

$$v(x) = v_{s,S}(x) = \begin{cases} S - x, & \text{if } x \leq s \\ 0, & \text{if x>s} \end{cases}$$

The feedback is used to compute an impulse control $V = V_{s,S}$ as follows: At any time t if $x(t) \leq s$ then order a quantity

$$v(t) = S - x(t).$$

If

$$x(t) > s,$$

then do not put an order at time t. We call

$$u_{s,S}(x).$$

The value function associated to a pair s, S. Namely

$$u_{s,S}(x) = J_x(V_{s,S}).$$

14.1.3. INEQUALITIES. It is easy to check inequalities verified by $u(x)$. Recall that x represents the state (the Inventory level) at initial time. If we force a decision to order a quantity v at initial time, then we pay $K + cv$ and the state becomes $x + v$. If we proceed optimally from now on, the best that we can achieve is

$$K + cv + u(x + v).$$

Therefore we can assert that

$$u(x) \leq K + \inf_{v>0}(cv + u(x + v)). \tag{14.1.1}$$

The next inequality consists in examining the decision not to order at initial time. So we delay the decision during a small amount of time δ. During this period we first pay a holding cost δxh, then we arrive at time δ with an inventory

$$x - \delta\lambda.$$

This number is positive for small δ. Suppose that we behave optimally from now on. The best that we can obtain for the life time (δ, ∞) is

$$u(x - \delta\lambda).$$

We discount this quantity to get a value at time 0. Since δ is small, we have the approximate value

$$(1 - \delta\alpha)u(x - \delta\lambda).$$

So we claim

$$u(x) \leq (1 - \delta\alpha)u(x - \delta\lambda) + \delta xh$$

Expanding in δ we obtain

$$\alpha u + \lambda u' \leq hx, x > 0 \tag{14.1.2}$$

The next point is to claim that for any initial inventory x, either (14.1.1) or (14.1.2) hold, since either we decide to order or we leave the system evolve freely for a small amount of time. We can express this by writing a complementary slackness condition

$$(\alpha u + \lambda u' - hx)(u(x) - K - \inf_{v>0}(cv + u(x + v))) = 0 \forall x > 0 \tag{14.1.3}$$

We must complete the conditions by considerations on the boundary and about smoothness. The boundary is $x = 0$ We clearly must order at the boundary, otherwise the inventory remains at 0, since it cannot become negative. Eventually we order and we can consider that the problem starts at this time. Therefore, we can write

$$u(0) = K + \inf_{v>0}(cv + u(v)). \tag{14.1.4}$$

Concerning smoothness, equation (14.1.3) is written without ambiguity if

$$u \in C^1(0, \infty). \tag{14.1.5}$$

The set of inequalities and complementarity slackness equation (14.1.1), (14.1.2), (14.1.3) and associated boundary and smoothness conditions is called a *Quasi-Variational Inequality*.

Remark 14.1. The smoothness is not guaranteed by the definition of the value function $u(x)$. It is a desirable property, which is far from automatic, in particular at switching points.

Let us now write conditions satisfied by a value function defined by an s, S policy. Note that

$$0 \leq s \leq S.$$

We first have

$$u_{s,S}(x) = K + c(S - x) + u_{s,S}(S), \forall x \leq s. \tag{14.1.6}$$

Next

$$\alpha u_{s,S} + \lambda u'_{s,S} - hx = 0, \forall x > s. \tag{14.1.7}$$

These relations express the decision making under an s, S policy.

Comparison with the relations defining the value function u imply a natural way to obtain S from s. Let us choose first a single positive number s and define $u_s(x)$ by

$$u_s(x) + cx = K + \inf_{y>s}(cy + u_s(y)), x \leq s \tag{14.1.8}$$

$$\alpha u_s + \lambda u'_s - hx = 0, \forall x > s. \tag{14.1.9}$$

Then $S(s)$ is obtained simply by

$$\inf_{y>s}(cy + u_s(y)) = cS + u_s(S) \tag{14.1.10}$$

14.1.4. DEFINITION OF u_s. For any fixed s, we can write the solution of the differential equation (14.1.9) as follows

$$u_s(x) = u_s(s) \exp -\frac{\alpha}{\lambda}(x - s) + \frac{h}{\lambda}\int_s^x y \exp -\frac{\alpha}{\lambda}(x - y)dy,$$

hence

$$u_s(x) = \frac{h}{\alpha}\left(x - \frac{\lambda}{\alpha}\right) + \left(u_s(s) - s\frac{h}{\alpha} + \frac{\lambda h}{\alpha^2}\right) \exp -\frac{\alpha}{\lambda}(x - s). \tag{14.1.11}$$

To define $S = S(s)$ we write

$$u'_s(S) = -c,$$

therefore

$$\alpha u_s(S) = hS + \lambda c.$$

We can write the following equation for S

$$\exp -\frac{\alpha}{\lambda}(S-s) = \frac{1+\frac{\alpha c}{h}}{1+\frac{\alpha}{\lambda}(\frac{\alpha}{h}u_s(s)-s)}.$$

To define the value $u_s(s)$ we use equations (14.1.8) and (14.1.10) to obtain

$$u_s(s) = K + c(S-s) + \frac{hS+\lambda c}{\alpha}.$$

We deduce the right derivative

$$u'_{s+0}(s) = -c - \frac{\alpha K + (\alpha c + h)(S-s)}{\lambda}.$$

Since

$$u'_{s-0}(s) = -c.$$

There is no way to have a matching of derivatives at point s, which we require if we want $u_s(x)$ to coincide with the value function $u(x)$.

The only way to avoid the difficulty is to take $s = 0$, since there is no matching on the boundary. Consider then $u_0(x)$. The value of $S = S(0)$ is obtained by solving the equation

$$\left(1+\frac{\alpha c}{h}\right)\exp\frac{\alpha}{\lambda}S - S\left(\frac{\alpha}{\lambda}+\frac{\alpha^2 c}{\lambda h}\right) - 1 - \frac{\alpha c}{h} - \frac{\alpha^2 K}{\lambda h} = 0. \tag{14.1.12}$$

It is easy to check that this equation has a unique positive solution S.

The important question now is to check whether $u_0(x)$ is the value function $u(x)$. We must verify that it satisfies the conditions for u, namely relations (14.1.1), (14.1.2), (14.1.3). The only non trivial verification is (14.1.1), which we rewrite as

$$u_0(x) + cx \leq K + \inf_{y>x}(u_0(y)+cy). \tag{14.1.13}$$

Since

$$u_0(x) = \frac{h}{\alpha}x - \frac{\lambda h}{\alpha^2} + \left(u_0(0)+\frac{\lambda h}{\alpha^2}\right)\exp -\frac{\alpha x}{\lambda},$$

we see easily that it is a convex function. Hence $u_0(x) + cx$ is also convex. Its minimum is at S. It is thus increasing for $x \geq S$. Hence for $x \geq S$ we have

$$\inf_{y>x}(u_0(y)+cy) = u_0(x) + cx,$$

and the property (14.1.13) is trivially verified.

For $x < S$ we can write

$$u_0(x) + cx \leq u_0(0).$$

Also

$$\inf_{y>x}(u_0(y)+cy) = u_0(S) + cS.$$

Noting that

$$u_0(0) = K + u_0(S) + cS,$$

the property (14.1.13) is obtained again.

14.2. ERGODIC PROBLEM

The ergodic problem consists in letting the discount tend to 0. We indicate the dependence in α for all quantities of interest. Denote in particular by $u_\alpha(x)$ the value function, recalling that

$$u_\alpha(x) = u_{\alpha 0}(x).$$

We omit the index 0 which is no more necessary. From direct checking

$$u_\alpha(x) \to \infty, \text{ as } \alpha \to 0$$

14.2.1. RECOVERING EOQ FORMULA.

Exercise 14.1. Show that

$$S_\alpha \to \hat{S} = \sqrt{\frac{2K\lambda}{h}} = \hat{q},$$

as $\alpha \to 0$, where $\hat{q}$ has been referred as the EOQ formula, see Chapter 2, 2.2.

We deduce that

$$\alpha u_\alpha(0) \to h\hat{S} + \lambda c,$$

which is the optimal average cost derived from the EOQ formula. Similarly

$$\alpha u_\alpha(x) \to h\hat{S} + \lambda c,$$

and we see that the initial inventory does not play any role.

Next set

$$\bar{u}_\alpha(x) = u_\alpha(x) - u_\alpha(0).$$

It satisfies the differential equation

$$\alpha u_\alpha(0) + \alpha \bar{u}_\alpha(x) + \lambda \bar{u}_\alpha'(x) = hx,$$

with the initial condition

$$\bar{u}_\alpha(0) = 0.$$

We know that

$$\alpha u_\alpha(0) \to h\hat{S} + \lambda c = \beta,$$

so we can conjecture that

$$\bar{u}_\alpha(x) \to u(x),$$

where u satisfies

$$\beta + \lambda u' = hx, \;\; 0(0) = 0,$$

i.e

$$u(x) = -\frac{\beta x}{\lambda} + \frac{x^2 h}{2\lambda}$$

Exercise 14.2. Check directly the limit property stated above.

14.2.2. EQUIVALENCE OF LIMITS.

Exercise 14.3. Consider a $0, q$ policy and set

$$q = \lambda T.$$

We start with an initial inventory 0, so we order right away. The inventory is defined

$$y(t) = q - \lambda(t - nT), \forall t \in [nT, (n+1)T].$$

Consider the average cost

$$C(q) + c\lambda = \frac{hq}{2} + \frac{K\lambda}{q} + c\lambda.$$

Define the discounted cost

$$J_\alpha(q) = h \int_0^\infty y(t) \exp -\alpha t dt + \sum_{n=0}^\infty (K + cq) \exp -n\alpha T.$$

Exercise 14.4. Show that

$$\alpha J_\alpha(q) \to C(q) + c\lambda,$$

as $\alpha \to 0$

Exercise 14.5. Compute the quantity $\hat{q}_\alpha$ which minimizes the quantity $J_\alpha(q)$. Show that it is the same as the one obtained by solving equation (14.1.12).

14.3. CONTINUOUS RATE DELIVERY

14.3.1. TRANSFORMED EOQ FORMULA. Suppose now that there is no immediate delivery. Instead, the delivery is provided at a continuous rate r. Since there is no possibility of shortage, we must have $r \geq \lambda$. We recall that the EOQ formula must be changed (see 14.3.1) into

$$\hat{q} = \sqrt{\frac{2K\lambda r}{h(r - \lambda)}}.$$

This formula leads to ∞ when $r = \lambda$. One must interpret it in the sense that there will be a continuous delivery at the level of the demand. The stock remains 0, and the cost per unit of time equal to $c\lambda$. There is no optimization in this case.

14.3.2. TRANSFORMED INEQUALITIES. We may attempt to derive the inequalities governing the value function for the problem with continuous delivery. We decide not to order while a delivery is processed. The value function $u(x)$ is considered with an initial inventory x and no delivery being processed. Inequality (14.1.2) is unchanged. However, inequality (14.1.1) must be changed to express the fact that when we order a quantity q, then we have to wait till it is delivered, which means till $\frac{q}{r}$. We have to cover the holding cost during this period and when the delivery is completed then the available inventory is

$$x + q - \frac{\lambda}{r} q.$$

So we must write

(14.3.1)

$$u(x) \leq K + \inf_{q>0} \left[cq + h \int_0^{\frac{q}{r}} (x + (r - \lambda)t) \exp -\alpha t dt + \exp -\alpha \frac{q}{r} \, u\left(x + q - \frac{\lambda}{r} q\right) \right]$$

Performing calculations we obtain

$$(14.3.2)\quad u(x) \le K + \inf_{q>0}\Bigg[cq + \frac{h}{\alpha}\left(x\left(1-\exp -\alpha\frac{q}{r}\right) + \frac{r-\lambda}{\alpha}\right.$$
$$\left.\cdot\left(1-\exp -\alpha\frac{q}{r} - \alpha\frac{q}{r}\exp -\alpha\frac{q}{r}\right)\right) + \exp -\alpha\frac{q}{r}u\left(x+q-\frac{\lambda}{r}q\right)\Bigg].$$

We proceed as in the EOQ case and consider $s = 0$ and that the equation is verified for $x > 0$ and use the boundary condition

$$(14.3.3)\qquad u(0) = K + \inf_{q>0}\Bigg[cq + \frac{h(r-\lambda)}{\alpha^2}\left(1-\exp -\alpha\frac{q}{r} - \alpha\frac{q}{r}\exp -\alpha\frac{q}{r}\right)$$
$$+\exp -\alpha\frac{q}{r}\,u(q-\frac{\lambda}{r}q)\Bigg].$$

Exercise 14.6. We compute the optimal $\hat{S} = \hat{q}$ by differentiating the right hand side and equaling it to 0. We get

$$(14.3.4)\quad c + \frac{r-\lambda}{r}\exp -\alpha\frac{S}{r}\left(\frac{h}{r}S + u'\left(S\frac{r-\lambda}{r}\right)\right) - \frac{\alpha}{r}\exp -\alpha\frac{S}{r}u\left(S\frac{r-\lambda}{r}\right) = 0$$

Exercise 14.7. Show that $\hat{S}$ is a solution of

$$(14.3.5)\quad \left(\exp -\frac{\alpha}{r}S + \frac{\alpha c}{h}\right)\left(\exp\frac{\alpha}{\lambda}S - 1\right) - \frac{r}{\lambda}\left(1-\exp -\frac{S\alpha}{r}\right) - S\frac{\alpha^2 c}{\lambda h} - \frac{\alpha^2 K}{\lambda h} = 0$$

Exercise 14.8. Check that equation (14.3.5) reduces to (14.1.12) as $r \to \infty$, and to the transformed EOQ formula of section 14.3.1 as $\alpha \to 0$.

14.4. LEAD TIME

14.4.1. NO SHORTAGE ADMITTED. We complement section 2.6.1.

We consider now a situation with discount rate of time α. If we implement an s, q policy with $s = \lambda L$, the optimal q will be different from the EOQ formula. Indeed, we recall the differential equation for $u_s(x)$ with $s = \lambda L$, which we write $u(x)$ to simplify the notation, although it is not the value function. We have

$$u(x) = \frac{h}{\alpha}\left(x - \frac{\lambda}{\alpha}\right) + \left(u(\lambda L) - \frac{L\lambda h}{\alpha} + \frac{\lambda h}{\alpha^2}\right)\exp -\frac{\alpha}{\lambda}(x - L\lambda),$$

then we must write

$$u'(\hat{q}) = -c\exp\alpha L,$$

where we expect $\hat{q} > L\lambda$. We can then apply the equation to obtain

$$u(\hat{q})\exp -\alpha L = \frac{\lambda c + h\hat{q}\exp -\alpha L}{\alpha}.$$

We deduce

$$u(\lambda L) = K + c\hat{q} + \frac{L\lambda h}{\alpha} - \frac{\lambda h}{\alpha^2} + \frac{h\lambda}{\alpha^2}\exp -\alpha L + \frac{\lambda c + h\hat{q}\exp -\alpha L}{\alpha}.$$

Collecting results, we obtain that $\hat{q}$ is the solution of the equation

(14.4.1)
$$\left(\exp -\alpha L + \frac{\alpha c}{h}\right)\exp\frac{\alpha}{\lambda}q - \frac{\alpha}{\lambda}q\left(\exp -\alpha L + \frac{\alpha c}{h}\right) - \exp -\alpha L - \frac{\alpha c}{h} - \frac{\alpha^2 K}{\lambda h} = 0.$$

Exercise 14.9. Check that when α is small, the solution of equation (14.4.1) converges to the EOQ formula, independent of L, provided it is larger than λL.

14.5. NEWSVENDOR PROBLEM

We consider here a period T. At the beginning of the period, the inventory is S, the quantity procured. The total demand in the period is D assumed to be random with a probability density function $f(x)$. It materializes during the period at a uniform rate $\frac{D}{T}$ So the inventory at time t is

$$S - \frac{D}{T}t.$$

This random variable can be positive or negative. When it is positive, it carries a holding cost. When it is negative it carries a shortage cost.

The average inventory surplus is

$$\begin{aligned} I_+(S) &= E\frac{\int_0^T (S - D\frac{t}{T})^+ dt}{T} = E\left[\mathbb{1}_{S>D}(S - \frac{D}{2}) + \mathbb{1}_{S\leq D}\frac{S^2}{2D}\right] \\ &= SF(S) - \frac{1}{2}\int_0^S xf(x)\,dx + \frac{S^2}{2}\int_S^{+\infty}\frac{f(x)}{x}\,dx. \end{aligned}$$

For the shortage inventory, we have

$$I_-(S) = E\frac{\int_0^T (S - D\frac{t}{T})^- dt}{T} = E\frac{((S-D)^-)^2}{2D},$$

which is

$$I_-(S) = I_+(S) - S + \frac{1}{2}\int_0^{+\infty} xf(x)\,dx.$$

We denote by h the holding cost per unit of time and by p the shortage cost per unit of time, so the total cost is

$$C(S) = hI_+(S) + pI_-(S),$$

or

$$\begin{aligned} C(S) = (h+p)\left[SF(S) - \frac{1}{2}\int_0^S xf(x)\,dx + \frac{S^2}{2}\int_S^{+\infty}\frac{f(x)}{x}\,dx\right] \\ +p\left(-S + \frac{1}{2}\int_0^{+\infty} xf(x)\,dx\right). \end{aligned}$$

We can then compute the optimal $S = \hat{S}$. We have

$$C'(S) = (h+p)\left(F(S) + S\int_S^{+\infty}\frac{f(x)}{x}\,dx\right) - p,$$

and

$$C\text{''}(S) = (h+p)\int_S^{+\infty}\frac{f(x)}{x}\,dx > 0.$$

Therefore the unique $\hat{S}$ is given by

$$F(\hat{S}) + \hat{S}\int_{\hat{S}}^{+\infty}\frac{f(x)}{x}\,dx = \frac{p}{h+p}$$

Remark 14.2. Usually the solution of the Newsvendor problem is given as

$$F(\hat{S}) = \frac{p}{h+p}.$$

This is because one assumes that the demand materializes in one shot.

14.6. POISSON DEMAND

14.6.1. POISSON PROCESS. Instead of modeling the randomness of the demand through a random rate λ, we shall consider the demand $D(t)$ to be a Markov process. In the deterministic case it is simply λt. The most natural stochastic process is the Poisson process with parameter λ. We recall the definition of the Poisson process. We have

$$D(0)=0;\ D(t_2)-D(t_1) \text{ and } D(s_2)-D(s_1) \text{ are independent } \forall s_1\leq s_2\leq t_1\leq t_2$$

$$D(t_2)-D(t_1) \text{ is integer valued; } P(D(t_2)-D(t_1)=k)=\exp -\lambda(t_2-t_1)\frac{(\lambda(t_2-t_1))^k}{k!}$$

The following formula expresses the Markov property

$$\begin{aligned}P(D(t+\epsilon)=D(t)|D(s),s\leq t)&\sim 1-\lambda\epsilon\\ P(D(t+\epsilon)=D(t)+1|D(s),s\leq t)&\sim \lambda\epsilon\end{aligned}$$

The Poisson process models the arrivals of customers at a queue, with average arrival rate λ. Note that

$$ED(t)=\lambda t.$$

This last property explains why the Poisson is a natural extension of the deterministic model with constant demand rate λ.

The infinitesimal generator of the Markov process $D(t)$ is the linear operator on smooth functions of the real variable x defined by

$$A\phi(x)=\lim_{\epsilon\to 0}\frac{\phi(x+D(\epsilon))-\phi(x)}{\epsilon}=\lambda(\phi(x+1)-\phi(x))$$

14.6.2. DESCRIPTION OF THE MODEL. An impulse control is a sequence

$$\theta_n, v_n,$$

where θ_n is a stopping time with respect to the filtration

$$\mathcal{F}^t=\sigma(D(s),s\leq t),$$

and v_n is a random variable $\mathcal{F}^{\theta_n}$ measurable. Denoting by V an impulse control, the corresponding inventory is described by the formula

$$y_x(t;V)=x-D(t)+M(t;V),$$

with

$$M(t)=M(t;V)=\sum_{\{n|\theta_n<t\}}v_n$$

Note that the inventory can become negative. Let

$$f(x)=hx^++px^-,$$

the cost function to be minimized is given by

$$\begin{aligned}J_x(V)=E\Bigg[&\sum_{n=0}^{\infty}(K+cv_n)\exp -\alpha\theta_n\\ &+\int_0^{\infty}f(y(t))\exp -\alpha t dt\Bigg]\end{aligned}$$

14.6.3. INEQUALITIES FOR THE VALUE FUNCTION. Considering the value function

$$u(x) = \inf_V J_x(V).$$

We can derive inequalities and complementarity slackness conditions for the function $u(x)$, which will be referred again as the *Quasi-Variational Inequality.* We first have

$$u(x) \leq K \inf_{v>0}(cv + u(x+v)), \tag{14.6.1}$$

then

$$-\lambda(u(x-1) - u(x)) + \alpha u(x) \leq f(x). \tag{14.6.2}$$

The complementarity slackness condition expresses

$$[((\alpha+\lambda)u(x) - \lambda u(x-1)) - f(x)][u(x) - K - \inf_{v>0}(cv + u(x+v))] = 0. \tag{14.6.3}$$

14.6.4. VALUE FUNCTION CORRESPONDING TO AN s, S POLICY. Note that $x \in R$. There is no explicit boundary condition. However $u(x)$ should not have a growth faster that $f(x)$ (namely linear growth). In a way similar to that discussed in the deterministic case, the value function corresponding to an s, S policy is defined as follows. We write

$$u_s(x) + cx = u_s(s) + cs = K + \inf_{\eta>s}(c\eta + u(\eta)) = K + cS + u(S), \forall x \leq s \tag{14.6.4}$$

$$(\alpha+\lambda)u(x) - \lambda u(x-1) - f(x) = 0, \forall x \geq s \tag{14.6.5}$$

Theorem 14.1. *Assume*

$$p > \alpha c.$$

There exists a unique pair s, S and a unique continuous function $u_s(x)$ with linear growth such that equations (14.6.4), (14.6.5) *are satisfied.*

PROOF. Noting that

$$u_s(s-1) + c(s-1) = u_s(s) + cs,$$

and

$$(\alpha+\lambda)u_s(s) - \lambda u_s(s-1) = f(s),$$

we get

$$u_s(x) + cx = cs + \frac{f(s) + \lambda c}{\alpha}, \forall x \leq s.$$

Define

$$H_s(x) = u_s(x) + cx - cs - \frac{f(s) + \lambda c}{\alpha}$$

We get the following relations for $H_s(x)$

$$(\alpha+\lambda)H_s(x) - \lambda H_s(x-1) = g(x) - g(s), \forall x \geq s; \quad H_s(x) = 0, \forall x \leq s, \tag{14.6.6}$$

where we have set

$$g(x) = f(x) + \alpha cx.$$

We have

$$g(x) - g(s) = \begin{cases} -(p-\alpha c)(x-s), & \forall x < 0 \\ (h+\alpha c)x + (p-\alpha c)s, & \forall x \geq 0 \end{cases}$$

When $s<0$, we see in particular, thanks to the assumption, that

$$g(x)-g(s)\leq 0, \forall s<x<0.$$

In fact, noting that there exists a unique $\nu>0$ such that $g(\nu)=g(s)$. We have

$$s<x<\nu \Rightarrow g(x)-g(s)<0.$$

Define

$$g_s(x)=(g(x)-g(s))\mathbb{1}_{x>s}.$$

We obtain

$$H_s(x)=\sum_{j=0}^{[x-s]}\frac{g_s(x-j)\lambda^j}{(\alpha+\lambda)^{j+1}}, \forall x\geq s; \quad H_s(x)=0, \forall x\leq s, \tag{14.6.7}$$

in which $[x]$ is the integer part of x, the unique integer such that

$$[x]\leq x<[x]+1.$$

Thanks to the fact that $g_s(s)=0$, we can check that $H_s(x)$ is a continuous function. The quantity $S=S(s)$ is obtained by

$$H_s(S)=\inf_{\eta>s}H_s(\eta).$$

Recalling the definition of $f(x)$ we get at once

$$S(s)=s, \forall s\geq 0$$

Consider next the case $s<0$. From formula (14.6.7) it follows immediately that

$$s<x<\nu \Rightarrow H_s(x)<0.$$

We can next write for $x>\nu$

$$H_s(x)=\sum_{j=0}^{[x-\nu]}\frac{g_s(x-j)\lambda^j}{(\alpha+\lambda)^{j+1}}+\sum_{j=[x-\nu]+1}^{[x-s]}\frac{g_s(x-j)\lambda^j}{(\alpha+\lambda)^{j+1}}.$$

The first sum is composed of positive terms, and the second of negative terms.

We deduce easily the following estimate

$$H_s(x)\geq \frac{g_s(x)}{\alpha+\lambda}+\frac{g_s(0)}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x-\nu]},$$

from which it follows that

$$H_s(x)\to +\infty \text{ as } x\to +\infty.$$

Recalling the continuity of $H_s(x)$ it follows that it reaches its minimum on (s,∞). Picking the smallest minimum $S(s)$ we define the function $S(s)$ for $s<0$. Necessarily

$$S(s)\geq 0,$$

since $H_s(x)$ decreases on $s<x<0$. This follows from direct checking of formula (14.6.7).

There remains to find s. It must satisfy the equation in s

$$\inf_{\eta>s}H_s(\eta)=-K.$$

We have

$$\inf_{\eta>s}H_s(\eta)=0, \forall s\geq 0.$$

So we may assume $s < 0$. Consider $s < x_0 < 0$. We can assert that

$$H_s(x_0) \leq \frac{g_s(x_0)}{\alpha + \lambda}.$$

Therefore

$$\inf_{\eta > s} H_s(\eta) \leq H_s(x_0) \leq \frac{g_s(x_0)}{\alpha + \lambda}.$$

Letting s go to $-\infty$, we obtain

$$\inf_{\eta > s} H_s(\eta) \to -\infty, \text{ as } s \to -\infty$$

Finally we show that $\inf_{\eta > s} H_s(\eta)$ is an increasing function of s on $(-\infty, 0)$. That will prove that there exists one and only one value of s such that this function equals $-K$. Consider $s' < s < 0$. We are going to prove that

$$H_{s'}(x) < H_s(x), \forall s' < s < x.$$

This will imply the desired result. We note that

$$g_{s'}(x) < g_s(x), \forall x.$$

Writing

$$H_{s'}(x) - H_s(x) = \sum_{j=0}^{[x-s]} \frac{(g_{s'}(x-j) - g_s(x-j))\lambda^j}{(\alpha+\lambda)^{j+1}} + \sum_{j=[x-s]+1}^{[x-s']} \frac{g_{s'}(x-j)\lambda^j}{(\alpha+\lambda)^{j+1}}.$$

The first term is negative from the property mentioned just above. In the second term we note that

$$[x-s] + 1 \leq j \leq [x-s'] \Rightarrow s' \leq x - j < s < 0,$$

and the second term is also negative. The property is proved, and s is defined uniquely. □

14.6.5. OPTIMALITY OF AN s, S POLICY. We now turn to proving that the function $u_s(x)$ defined above with the choice of s is indeed the optimal policy, i.e. that $u_s(x) = u(x)$.

Theorem 14.2. *We assume $c\alpha - p < 0$. Then, the s, S policy is optimal.*

Proof. We have to verify the inequalities, the complementarity slackness condition being trivially satisfied. The inequality

$$(\alpha + \lambda)u(x) - \lambda u(x-1) \leq f(x), \forall x \leq s,$$

amounts to $g(x) - g(s) \geq 0, \forall x \leq s$. Since $s < 0$, this property is satisfied. The second inequality reduces to

$$H_s(x) \leq K + \inf_{\eta > x} H_s(\eta), \forall x \geq s$$

Since $K = -\inf_{\eta > s} H_s(\eta)$ the right hand side is certainly positive. So the inequality is obvious whenever $H_s(x) \leq 0$. This occurs in particular for $s < x < \nu$. The function $H_s(x)$ can be negative beyond ν and there exists a first point x_0 with $\nu < x_0$ and $H_s(x_0) = 0$. So it is sufficient to prove the inequality for $x \geq x_0$. We also note that

$$g_s(x_0) = -\lambda H_s(x_0 - 1) \geq 0.$$

Let $\xi \geq 0$. We shall consider $x > x_0 - \xi$. Since ξ is arbitrary, we may have $x < s$. We can write the equation for $H_s(x)$ in a way which is valid for any x as follows (recalling the definition of $g_s(x)$)

$$(\alpha+\lambda)H_s(x) - \lambda H_s(x-1) = g_s(x).$$

We can next write the following inequalities

$$(\alpha+\lambda)H_s(x) - \lambda H_s(x-1)\mathbb{1}_{x-1\geq x_0-\xi} \leq g_s(x),$$
$$H_s(x_0-\xi) \leq 0.$$

The reason of the truncation is to be able to work on the domain $x \geq x_0 - \xi$. The price to pay is that we can write only an inequality. Note Indeed that

$$H_s(x-1)\mathbb{1}_{x-1<x_0-\xi} \leq 0,$$

from the definition of x_0.

Define next

$$M_s(x) = H_s(x+\xi) + K \geq 0.$$

We can write

$$(\alpha+\lambda)M_s(x) - \lambda M_s(x-1)\mathbb{1}_{x-1\geq x_0-\xi} \geq g_s(x+\xi) + \alpha K,$$
$$M_s(x_0-\xi) = K.$$

We claim that

$$g_s(x+\xi) \geq g_s(x)$$

Note that

$$g_s(x+\xi) \geq g_s(x_0) > 0.$$

The claim is thus obvious if $x < s$. If $x > s$ then we have to prove that

$$g(x+\xi) \geq g(x).$$

If $x > 0$ this follows from convexity. If $s < x < 0$ then

$$g(x+\xi) - g(x) \geq g(x_0) - g(s) > 0.$$

Therefore setting

$$Y_s(x) = H_s(x) - M_s(x),$$

we deduce that $Y_s(x) \leq 0$. This result is valid for $x > x_0 - \xi$, hence also for $x > x_0$.

The proof of Theorem 14.2 has been completed. $\square$

14.6.6. COMPUTATION OF DERIVATIVE. We will compute the derivatives $H_s'(x+0)$ and $H_s'(x-0)$. For $x \geq s$ we have

$$H_s'(x+0) = \sum_{j=0}^{[x-s]} \frac{g_s'(x-j+0)\lambda^j}{(\alpha+\lambda)^{j+1}}$$

We can compute also $H_s'(x-0)$ by the formula

$$H_s'(x-0) = \begin{cases} \sum_{j=0}^{[x-s]} \frac{g_s'(x-j-0)\lambda^j}{(\alpha+\lambda)^{j+1}}, & \text{if } [x-s] < x-s \\ \sum_{j=0}^{[x-s]-1} \frac{g_s'(x-j-0)\lambda^j}{(\alpha+\lambda)^{j+1}}, & \text{if } [x-s] = x-s \end{cases}$$

with $H_s'(s-0) = 0$.

It follows that for $x < 0$

$$H_s'(x+0) = -\frac{p-\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x-s]+1}\right), \tag{14.6.8}$$

$$H'_s(x-0) = \begin{cases} H'_s(x+0), & \text{if } [x-s] < x-s \\ -\frac{p-\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{x-s}\right), & \text{if } [x-s] = x-s \end{cases} \tag{14.6.9}$$

It follows that for $x < 0$, $H'_s(x)$ is discontinuous only at points x such that $x - s$ is a negative integer. The discontinuity is

$$H'_s(x+0) - H'_s(x-0) = -\frac{p-\alpha c}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{x-s}.$$

If $s = [s]$ then the discontinuity points are negative integers $-n$ with $1 \le n \le -s$. We have

$$H'_s(-n+0) = -\frac{p-\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{-n-s+1}\right),$$

$$H'_s(-n-0) = -\frac{p-\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{-n-s}\right),$$

and for $-n < x < -n+1$

$$H'_s(x) = H'_s(-n+0).$$

Therefore for s negative integer, $x < 0$, $H'_s(x)$ is piece wise constant negative and decreasing with discontinuity points at values $x = -n$, $0 \le n \le -s$.

Assume next that $[s] < s$, note that $[s] + 1 = -[-s]$. We consider the intervals

$$(s, s+1), (s+1, s+2), \cdots, (s+[-s], 0).$$

An interval is of the form $(s+n-1, s+n)$ with n running from 1 to $[-s]+1$. For the last one the end $s+[-s]+1$ is replaced by 0. We have

$$s+n-1 < x < s+n \Rightarrow [x-s] = n-1,$$

and

$$H'_s(x) = H'_s(s+n-1+0) = -\frac{p-\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{n}\right).$$

We obtain that for $[s] < s$, $x < 0$, $H'_s(x)$ is piece wise constant negative and decreasing with discontinuity points at values $s+n-1$ with n running from 1 to $[-s]+1$.

Consider now $x \ge 0$, hence $[x] \ge 0$. We have $[x-s] \ge [x]$. We compute easily

$$H'_s(x+0) = \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x]+1}\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x]+1}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x-s]-[x]}\right). \tag{14.6.10}$$

The situation is more complex for $H'_s(x-0)$. We have 4 situations, depending on whether x and $x-s$ are integers or not. If $[x-s] < x-s$ and $[x] < x$ then

$$H'_s(x-0) = H'_s(x+0). \tag{14.6.11}$$

If $[x-s] < x-s$ and $[x] = x$ which implies $[x-s] = x + [-s]$ then

$$H_s'(x-0) = \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^x\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^x\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[-s]+1}\right). \tag{14.6.12}$$

Comparing with (14.6.10) we observe a discontinuity

$$H_s'(x+0) - H_s'(x-0) = \frac{1}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^x (h+p). \tag{14.6.13}$$

If $[x-s] = x-s$ then $[x] = x-s+[s]$. Note that in this case $x = [x]$ only if $s = [s]$. If $[s] < s$ we obtain

$$H_s'(x-0) = \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x]+1}\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x]+1}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[-s]}\right), \tag{14.6.14}$$

and if $[s] = s$ we obtain

$$H_s'(x-0) = \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^x\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^x\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{-s}\right) \tag{14.6.15}$$

We obtain again a discontinuity: When $[x-s] = x-s$ and $[s] < s$

$$H_s'(x+0) - H_s'(x-0) = -\frac{p-\alpha c}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{[x]-[s]}.$$

When $[x] = x$ and $[s] = s$ we get

$$H_s'(x+0) - H_s'(x-0) = \frac{1}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^x\left[h+\alpha c + (p-\alpha c)\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{-s}\right)\right].$$

We begin with the case $s = [s]$, where s is an integer. We have $[x-s] = [x]-s$. The only possible discontinuities are the positive integers. So we consider the intervals

$$(0,1), \cdots (n-1,n), \cdots .$$

If $n-1 < x < n$, we have

$$[x-s] = n-1-s < x-s,$$

therefore

$$\begin{aligned} H_s'(x) &= H_s'(n-1+0) \\ &= \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^n\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^n\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{-s}\right). \end{aligned}$$

So for $x \geq 0$ and s a negative integer, the function $H_s'(x)$ is piecewise constant, has discontinuity points at values x which are positive integers. The function $H_s'(x)$ is increasing.

Consider now $x \geq 0$ with $[s] < s$. Consider $n-1 < x < n$, then $[x] = n-1$ and

$$H_s'(n-1+0) = \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^n\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^n\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[-s]}\right),$$

$$H_s'(n-1-0) = \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{n-1}\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{n-1}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[-s]+1}\right).$$

Then for $n-1 < x < n+[-s]+s$ we have

$$[x-s] = n-1+[-s], \quad H_s'(x) = H_s'(n-1+0),$$

and for $n+[-s]+s < x < n$ we have

$$[x-s] = n+[-s], \quad [x-s]-[x] = [-s]+1$$

and

$$\begin{aligned} H_s'(x) &= \frac{h+\alpha c}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^n\right) - \frac{p-\alpha c}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^n\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[-s]+1}\right) \\ &= H_s'(n-0). \end{aligned}$$

We have a discontinuity at $n+[-s]+s$ given by

$$\begin{aligned} H_s'(n+[-s]+s+0) - H_s'(n+[-s]+s-0) &= H_s'(n-0) - H_s'(n-1+0) \\ &= -\frac{p-\alpha c}{\alpha+\lambda}\left(\frac{\lambda}{\lambda+\alpha}\right)^{n+[-s]}. \end{aligned}$$

So for $x \geq 0$, $[s] < s$ $H_s'(x)$ has discontinuity points at $x = n$ and $x = n+[-s]+s$, $n > 0$. It is piecewise constant, but not increasing. It increases at n and decreases at $n+[-s]+s$. However

$$H_s'(n+1+0) - H_s'(n+0) = \frac{1}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{n+1}\left[h+\alpha c+(p-\alpha c)\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{[-s]}\right)\right]$$

The minimum $S(s)$ must satisfy $H_s'(S+0) \geq 0$. Therefore S is necessarily an integer. If s is an integer, we deduce

$$H_s'(x+0) \geq 0, \forall x > S.$$

Hence

$$H_s(x) = \inf_{\eta > x} H_s(\eta), \forall x \geq S.$$

Now for $s \leq x \leq S$ we have

$$H_s(x) \leq 0,$$

and

$$\inf_{\eta > x} H_s(\eta) = H_s(S) = -K$$

If s is not an integer, we can only deduce

$$H_s'(n+1+0) \geq H_s'(n+0) \geq 0, \forall n > S.$$

We may have some negative values of the derivative. This can occur at intervals $(n+[-s]+s, n)$. However, we know that for $x \geq s$

$$H_s(x) \leq K + H_s(x+\xi), \ \forall \xi \geq 0$$

14.6.7. OBTAINING S. The number S is characterized by the fact that it is the first number strictly larger than s such that

$$H_s'(S-0) \leq 0, H_s'(S+0) \geq 0.$$

Since $H_s'(x)$ is piecewise constant it is the first integer n such that

$$H_s'(n+0) = \frac{1}{\alpha}\left[(\alpha c+h)\left(1-\left(\frac{\lambda}{\alpha+\lambda}\right)^{n+1}\right)\right.$$
$$\left. -(p-\rho c)\left(\frac{\lambda}{\rho+\lambda}\right)^{n+1}\left(1-\left(\frac{\lambda}{\rho+\lambda}\right)^{[-s]}\right)\right] \geq 0,$$

and

$$H_s'(n-1+0) = \frac{1}{\rho}\left[(\rho c+h)\left(1-\left(\frac{\lambda}{\rho+\lambda}\right)^{n}\right)\right.$$
$$\left. -(p-\rho c)\left(\frac{\lambda}{\rho+\lambda}\right)^{n}\left(1-\left(\frac{\lambda}{\rho+\lambda}\right)^{[-s]}\right)\right] < 0.$$

14.7. ERGODIC CASE FOR THE POISSON DEMAND

The objective now is to let α tend to 0. In this case

$$g_s(x) = (f(x)-f(s))1\!\!1_{x>s}.$$

Next the function $H_s(x)$ becomes

$$H_s(x) = \sum_{j=0}^{[x-s]} \frac{g_s(x-j)}{\lambda}, \forall x \geq s; \quad H_s(x) = 0, \forall x \leq s \tag{14.7.1}$$

The number $S(s)$ is obtained by computing the left and right derivatives

$$H_s'(x+0) = \sum_{j=0}^{[x-s]} \frac{g_s'(x-j+0)}{\lambda}$$

$$H_s'(x-0) = \begin{cases} \sum_{j=0}^{[x-s]} \frac{g_s'(x-j-0)}{\lambda}, & \text{if } [x-s] < x-s \\ \sum_{j=0}^{[x-s]-1} \frac{g_s'(x-j-0)}{\lambda}, & \text{if } [x-s] = x-s \end{cases}$$

and we write

$$H_s'(S+0) \geq 0; \quad H_s'(S-0) \leq 0$$

Lemma 14.1. *Assume $-s$ to be integer and*

$$-\frac{ps}{h} \textit{ not integer,}$$

then

$$S(s) = \left[-\frac{ps}{h}\right].$$

PROOF. Note that

$$[S-s] = [S]-s.$$

Recalling that

$$g_s'(x+0) = \begin{cases} h, & \text{if } x \geq 0 \\ -p, & \text{if } x < 0 \end{cases}$$

and

$$g_s'(x-0) = \begin{cases} h, & \text{if } x > 0 \\ -p, & \text{if } x \leq 0 \end{cases}$$

The first condition implies

$$[S] + 1 \geq -\frac{ps}{h}.$$

The equality is not possible, because of the assumption hence

$$[S] + 1 > -\frac{ps}{h}.$$

For the second condition we have to consider the case $S - s$ integer or not integer. If it is not an integer then S cannot be integer, but then $H_s'(S-0) = H_s'(S+0) = 0$ which is not possible. If S is integer then the second condition amounts to

$$hS + ps \leq 0,$$

hence the result. □

We can then compute $H_s(S)$ by the formula

$$H_s(S) = \frac{1}{\lambda}\left[-p\frac{s^2}{2} + p\frac{s}{2} + S\left(ps + h\frac{S+1}{2}\right)\right].$$

To define the optimum s we must write the equation

$$K\lambda = p\frac{s^2}{2} - p\frac{s}{2} - S\left(ps + h\frac{S+1}{2}\right).$$

If we approximate S by $-p\frac{s}{h}$ we deduce easily that s is defined by

$$s = -\sqrt{\frac{2hK\lambda}{p(h+p)}}, \tag{14.7.2}$$

and the quantity ordered $S - s$ is given by

$$S - s = \sqrt{\frac{2K\lambda(p+h)}{ph}}. \tag{14.7.3}$$

We see that as $p \to \infty$ we obtain exactly the EOQ formulas.

LITTLE'S LAW:

Since we have Backlog and waiting time, a natural question is whether Little's Law applies to this model. Waiting starts between the moment the stock is zero till the time it becomes $s < 0$. If τ denotes this random time we have

$$\tau = \inf\{t | D(t) = -s\}.$$

To simplify we assume s integer. Then

$$D(\tau) = -s.$$

Since on $0, \tau$ we have

$$x(t) = -D(t),$$

the average backlog is

$$\frac{1}{\tau}\int_0^\tau D(t)dt,$$

and the average waiting time is

$$E\frac{1}{\tau}\int_0^\tau (\tau - t)dt = \frac{1}{2}E\tau.$$

Little's Law is the following relation

$$E\frac{1}{\tau}\int_0^\tau D(t)dt = \lambda\frac{1}{2}E\tau.$$

The left hand-side can be written as

$$E\int_0^1 D(\rho\tau)d\rho.$$

The result follows from the following property:

Exercise 14.10. Check the formula

$$ED(\tau) = \lambda E\tau,$$

for any stopping time with respect to the filtration $\mathcal{F}^t$ generated by the process $D(t)$.

The proof of the property goes as follows. Suppose

$$\tau = \nu h,$$

where ν is a random integer and h is a deterministic real number. Then

$$D(\nu h) = \sum_{j\geq 0}(D((j+1)h) - D(jh))\mathbb{1}_{jh<\nu h}.$$

Noting that

$$\mathbb{1}_{jh<\nu h} \text{ is } \mathcal{F}^{jh} \text{ measurable},$$

and $D((j+1)h) - D(jh)$ is independent from $\mathcal{F}^{jh}$, we deduce easily

$$ED(\nu h) = \sum_{j\geq 0} hE\mathbb{1}_{j<\nu} = \lambda hE\nu.$$

So the result is true for stopping times of the form νh. Now writing

$$ED\left(\left[\frac{\tau}{h}\right]h\right) \leq ED(\tau) \leq ED\left(\left[\frac{\tau}{h}\right]h + h\right),$$

it follows

$$\lambda E\left(\left[\frac{\tau}{h}\right]\right)h \leq ED(\tau) \leq \lambda E\left(\left[\frac{\tau}{h}\right]+1\right)h,$$

and the result follows.

14.8. POISSON DEMAND WITH LEAD TIME

14.8.1. DESCRIPTION OF THE MODEL. We introduce a deterministic lead-time L and we assume that *we do not order while waiting for a delivery.* This can be justified by logistics and security reasons. Therefore the model is transformed as follows:

$$y_x(t;V) = x - D(t) + M(t-L;V),$$

with

$$M(t;V) = \sum_{\{n|\theta_n<t\}} v_n,$$

and the impulse times θ_n verify

$$\theta_{n+1} > \theta_n + L.$$

The cost function is given by

$$J_x(V) = E\left[\sum_{n=0}^{\infty}(K + cv_n)\exp -\alpha\theta_n + \int_0^{\infty} f(y_x(t;V))\exp -\alpha t dt\right].$$

At time 0 we assume that there is no delivery in the pipe line.

14.8.2. INEQUALITIES FOR THE VALUE FUNCTION. The value function

$$u(x) = \inf_V J_x(V),$$

satisfies the following set of inequalities and complementarity slackness conditions (*Variational Inequality)*

$$\begin{aligned} u(x) \le K + E\int_0^L f(x - D(t))\exp -\rho t dt \\ + \inf_{v\ge 0}[cv + Eu(x + v - D(L))\exp -\alpha L] = M(u)(x), \end{aligned} \tag{14.8.1}$$

and

$$-\lambda(u(x-1) - u(x)) + \alpha u(x) \le f(x). \tag{14.8.2}$$

The complementarity slackness condition expresses

$$[((\alpha + \lambda)u(x) - \lambda u(x-1)) - f(x)][u(x) - M(u)(x)] = 0 \tag{14.8.3}$$

14.8.3. OPTIMAL s, S POLICY. The concept of s, S policy extends easily. It is convenient to introduce the function

$$g(x) = f(x) + \alpha cx + \lambda E\int_0^L (f(x - 1 - D(t)) - f(x - D(t)))\exp -\alpha t dt$$
$$-\alpha E\int_0^L f(x - D(t))\exp -\alpha t dt$$

and we set

$$g_s(x) = (g(x) - g(s))1\!\!1_{x>s}.$$

We solve as before, for a given s

$$(\alpha + \lambda)H_s(x) - \lambda H_s(x-1) = g_s(x), \forall x \ge s;\quad H_s(x) = 0, \forall x \le s \tag{14.8.4}$$

We define next the value function corresponding to s

$$u_s(x) = H_s(x) - cx + E\int_0^L f(x - D(t))\exp -\alpha t dt + C(s),$$

where $C(s)$ depends only on s. We want

$$(\alpha + \lambda)u_s(x) - \lambda u_s(x-1) = f(x), \forall x \ge s.$$

We check easily that the constant $C(s)$ must satisfy

$$\alpha C(s) = g(s) + \lambda c.$$

On the other hand

$$C(s) = K + \inf_{\eta \ge s}[c\eta + Eu_s(\eta - D(L))\exp -\alpha L].$$

We finally derive the equation to obtain the value of s

$$\frac{(g(s)+\lambda c)(1-\exp -\alpha L)}{\alpha} = K + c\lambda L \exp -\alpha L + \left\{ \inf_{\eta \geq s} c\eta(1-\exp -\alpha L) + \exp -\alpha L\, E[H_s(\eta - D(L)) + \int_0^L f(\eta - D(t+L)) \exp -\alpha t\, dt] \right\} \tag{14.8.5}$$

There is a useful transformation of the preceding relations, by making use of an important identity, namely

$$\exp -\alpha L\, Ef(x-D(L)) - f(x) + (\alpha+\lambda)\int_0^L f(x-D(t)) \exp -\alpha t dt - \lambda \int_0^L f(x-1-D(t)) \exp -\alpha t dt = 0, \forall x \tag{14.8.6}$$

Exercise 14.11. Show identity (14.8.6). Check that it reduces to the identity

$$-x^+ + \exp -\alpha L\, E(x-D(L))^+ + (\alpha+\lambda)E\int_0^L (x-D(t))^+ \exp -\alpha t dt - \lambda E \int_0^L (x-1-D(t))^+ \exp -\alpha t dt = 0.$$

The easiest way is to show that the derivative in x of this expression vanishes. Since it is 0 for $x \leq 0$, it vanishes for all x.

With this expression we can assert that

$$g(x) = \alpha c x + \exp -\alpha L\, Ef(x-D(L)), \tag{14.8.7}$$

so we get the

Lemma 14.2. *The function $g(x)$ is convex and continuous.*

PROOF. It follows immediately from the convexity of f. □

The function g is not continuously differentiable, and piece-wise linear.

Lemma 14.3. *Assume*

$$\alpha c - p \exp -\alpha L < 0,$$

then the function $g(x)$ satisfies

$$g(x) \to +\infty, \text{ as } |x| \to +\infty.$$

It has a unique minimum denoted $\hat{n}$ which is a positive integer.

PROOF. Note that

$$g'(x) = \alpha c - p \exp -\alpha L, \text{ if } x < 0,$$

and the assumption implies that $g(x)$ decreases for $x < 0$. Moreover it guaranties that

$$g(x) \to +\infty, \text{ as } x \to -\infty.$$

The same property holds trivially when $x \to +\infty$. Since the function is piece-wise linear with discontinuities at positive integers, the minimum is integer. It is a positive number, since the function decreases for $x < 0$. □

Writing explicitly

$$g(x) = (\alpha c - p\exp -\alpha L)x + \lambda L p \exp -\alpha L + (h+p)\exp -(\alpha+\lambda)L \sum_{j=0}^{[x]} (x-j)\frac{(\lambda L)^j}{j!}.$$

We deduce

$$g'(n+0) = \alpha c - p\exp -\alpha L + (h+p)\exp -(\alpha+\lambda)L \sum_{j=0}^{n} \frac{(\lambda L)^j}{j!}.$$

This is strictly increasing in n. Hence the minimum is unique.

We next transform the equation of which s is a solution, namely equation (14.8.5). We use a new identity

$$(14.8.8)\quad E\int_0^L g(x-D(t))\exp -\alpha t dt = cx(1-\exp -\alpha L) + \lambda c L \exp -\alpha L$$
$$-\lambda c \frac{1-\exp -\alpha L}{\alpha} + \exp -\alpha L E \int_0^L f(x - D(t+L))\exp -\alpha t dt$$

With this identity we can write equation (14.8.5) as

(14.8.9)

$$0 = K + \inf_{\eta \geq s} E\left[\exp -\alpha L H_s(\eta - D(L)) + \int_0^L (g(\eta - D(t)) - g(s))\exp -\alpha t dt\right]$$

Define

$$\psi_s(x) = E\left[\exp -\alpha L H_s(x - D(L)) + \int_0^L (g(x-D(t)) - g(s))\exp -\alpha t dt\right],$$

then we get the equation

$$0 = K + \inf_{\eta \geq s} \psi_s(\eta).$$

We can prove a new expression for $\psi_s(x)$. We have the

Lemma 14.4.

$$(14.8.10)\quad \psi_s(x) = H_s(x) + \exp -(\alpha+\lambda)L \sum_{j=[(x-s)^+]+1}^{\infty} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}}$$
$$\sum_{k=j+1}^{\infty} \frac{((\alpha+\lambda)L)^k}{k!}.$$

PROOF. We use the formulas

$$\int_0^L (g(x-D(t)) - g(s))\exp -\alpha t dt$$
$$= \exp -(\alpha+\lambda)L \sum_{j=0}^{\infty} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} \sum_{k=j+1}^{\infty} \frac{((\alpha+\lambda)L)^k}{k!},$$

and for $x \geq s$

$$\exp -\alpha L\; EH_s(x - D(L))$$
$$= \exp -(\alpha+\lambda)L \sum_{j=0}^{[x-s]} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} \sum_{k=0}^{j} \frac{((\alpha+\lambda)L)^k}{k!}.$$

To obtain the first one, one needs to compute the integral

$$A_j = \int_0^L \frac{(\lambda t)^j}{j!} \exp -(\alpha+\lambda)t dt.$$

One can check, by an induction argument, that

$$A_j = \frac{\lambda^j \exp -(\alpha+\lambda)L}{(\alpha+\lambda)^{j+1}} \sum_{k=j+1}^{\infty} \frac{((\alpha+\lambda)L)^k}{k!}.$$

The result is obtained by using the definition of $H_s(x)$.

Collecting results we can write, for $x \geq s$

$$\psi_s(x) = \sum_{j=0}^{[x-s]} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} + \exp -(\alpha+\lambda)L \sum_{j=[x-s]+1}^{\infty} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} \sum_{k=j+1}^{\infty} \frac{((\alpha+\lambda)L)^k}{k!}, \tag{14.8.11}$$

and for $x \leq s$

$$\psi_s(x) = \exp -(\alpha+\lambda)L \sum_{j=0}^{\infty} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} \sum_{k=j+1}^{\infty} \frac{((\alpha+\lambda)L)^k}{k!} \tag{14.8.12}$$

□

14.8.4. EXISTENCE OF $S(s)$. Consider s fixed. We shall show that $\psi_s(x)$ is bounded below and tends to $+\infty$ as $x \to +\infty$. From formula (14.8.11) recalling the definition of $\hat{n}$ we have

$$\psi_s(x) \geq \frac{g(\hat{n})-g(s)}{\alpha}.$$

Also

$$\psi_s(x) \geq \frac{g(x)-g(s)}{\alpha+\lambda} + \frac{\lambda}{\alpha}\frac{g(\hat{n})-g(s)}{\alpha+\lambda}.$$

So the minimum of $\psi_s(x)$ is attained and we define $S(s)$ as the smallest minimum.

14.8.5. EXISTENCE OF s. It remains to solve equation (14.8.9).

Theorem 14.3. *There exists a s solution of* (14.8.9) *such that $s < \hat{n}$. It is unique in the interval* $(-\infty, \hat{n}]$.

PROOF. Define

$$z(s) = \inf_{\eta \geq s} \psi_s(\eta).$$

Equation (14.8.9) means

$$z(s) + K = 0.$$

Note that

$$z(\hat{n}) \geq 0,$$

which follows immediately from the definition of $\psi_s(\eta)$. We are going to show that $z(s)$ is increasing for $s < \hat{n}$. Take

$$s' < s < \eta.$$

We have the formula

$$\psi_{s'}(\eta)-\psi_s(\eta)=\sum_{j=0}^{[\eta-s]}\frac{(g(s)-g(s'))\lambda^j}{(\alpha+\lambda)^{j+1}}+\sum_{j=[\eta-s]+1}^{[\eta-s']}\frac{(g(\eta-j)-g(s'))\lambda^j}{(\alpha+\lambda)^{j+1}}$$

$$-\exp-(\alpha+\lambda)L\sum_{j=[\eta-s]+1}^{[\eta-s']}\frac{(g(\eta-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}}\sum_{k=j+1}^{\infty}\frac{((\alpha+\lambda)L)^k}{k!}$$

$$+\exp-(\alpha+\lambda)L\sum_{j=[\eta-s']+1}^{\infty}\frac{(g(s)-g(s'))\lambda^j}{(\alpha+\lambda)^{j+1}}\sum_{k=j+1}^{\infty}\frac{((\alpha+\lambda)L)^k}{k!}.$$

Hence

$$g(s)-g(s')<0$$

Furthermore for $[\eta-s]+1\le j\le[\eta-s']$ we have

$$s\ge\eta-j\ge s'$$

Collecting results we can assert that

$$s'<s<\hat{n},s'<s<\eta\rightarrow\psi_{s'}(\eta)<\psi_s(\eta)$$

Therefore for $s'<s<\hat{n}$ we can assert that

$$z(s')\le\inf_{\eta\ge s}\psi_{s'}(\eta)\le\inf_{\eta\ge s}\psi_s(y)=z(s).$$

We next check that

$$z(s)\rightarrow-\infty \text{ as } s\rightarrow-\infty.$$

It is sufficient to show that

$$\psi_s(x)\rightarrow-\infty \text{ as } s\rightarrow-\infty,\forall x \text{ fixed}.$$

Recalling that for $x<0,s<0$

$$g(x)-g(s)=(\alpha c-p\exp-\alpha L)(x-s),$$

we may derive easily the inequality

$$\exp-(\alpha+\lambda)L\sum_{j=[x-s]+1}^{\infty}\frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}}\sum_{k=j+1}^{\infty}\frac{((\alpha+\lambda)L)^k}{k!}$$

$$\le(p\exp-\alpha L-\alpha c)\sum_{j=[x-s]+1}^{+\infty}\frac{j\lambda^j}{(\alpha+\lambda)^{j+1}}\rightarrow0 \text{ as } s\rightarrow-\infty$$

Consider next

$$\sum_{j=0}^{[x-s]}\frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}}$$

$$=\sum_{j=0}^{[x-\hat{n}]}\frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}}+\sum_{j=[x-\hat{n}]+1}^{[x-s]}\frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}}$$

The second term is positive. We use the fact that for $s<0$

$$g(x)-g(s)=(\alpha c-p\exp-\alpha L)(x-s)+(h+p)\exp-\alpha L\;E(x-D(L))^+,$$

and for $x>s$

$$g(x)-g(s)\le(h+p)\exp-\alpha L\;E(x-D(L))^+\le(h+p)\exp-\alpha L\;x^+$$

Therefore

$$\sum_{j=[x-\hat{n}]+1}^{[x-s]} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} \leq \frac{(h+p)\exp -\alpha L\, x^+}{\alpha}.$$

So we can assert

$$\limsup_{s\to-\infty} \psi_s(x) \leq \sum_{j=0}^{[x-\hat{n}]} \frac{(g(x-j)-g(s))\lambda^j}{(\alpha+\lambda)^{j+1}} + \frac{(h+p)\exp -\alpha L\, x^+}{\alpha}.$$

The first sum is composed of terms, each of them tends to $-\infty$ as $s \to -\infty$. This concludes the proof of Theorem 14.3. □

14.8.6. OPTIMALITY OF THE s, S POLICY. We have to check the set of inequalities and complementarity slackness conditions. The complementarity slackness property is clearly satisfied. When $x < s$ we have

$$u_s(x) = -cx + E\int_0^L f(x-D(t))\exp -\alpha t dt + \frac{g(s)+\lambda c}{\alpha},$$

and we must check that

$$(\alpha+\lambda)u_s(x) - \lambda u_s(x-1) \leq f(x).$$

Using the value of $u_s(x)$ this amounts to $g(x) \geq g(s)$ which is obviously satisfied since $s \leq \hat{n}$.

The difficult part is for $x \geq s$ where we must satisfy the inequality (14.8.1). Replacing with $H_s(x)$ this amounts to

$$H_s(x) + (g(s)+\lambda c)\frac{1-\exp -\alpha L}{\alpha} \leq K + c\lambda L\exp -\alpha L + \inf_{\eta\geq x}\Bigg\{c\eta(1-\exp -\alpha L)$$
$$+\exp -\alpha L\; E\left[H_s(\eta - D(L)) + \int_0^L f(\eta - D(t+L))\exp -\alpha t dt\right]\Bigg\},$$

which is interpreted as

$$H_s(x) \leq K + \inf_{\eta\geq x} \psi_s(\eta). \tag{14.8.13}$$

Since the right hand side is positive, this has to be checked only when $x > x_0$ first value for which $H_s(x)$ is positive or equal to 0.

Recalling the formula

$$H_s'(x+0) = \sum_{j=0}^{[x-s]} \frac{g'(x-j+0)\lambda^j}{(\alpha+\lambda)^{j+1}},$$

we note that $x_0 \geq \hat{n}$.

We can also write for x integer the formula

$$\left(\frac{\alpha+\lambda}{\lambda}\right)^x H_s'(x+0) = \sum_{k=-[-s]}^{x} \frac{g'(k+0)}{\alpha+\lambda}\left(\frac{\alpha+\lambda}{\lambda}\right)^k.$$

Consider then $x \geq x_0$, integer. We may assume $H_s'(x_0+0) \geq 0$, which corresponds to assuming $H_s(x) > 0$ slightly after x_0, otherwise the property we want to prove is valid beyond x_0.

Therefore

$$\sum_{k=-[-s]}^{x_0} \frac{g'(k+0)}{\alpha+\lambda}\left(\frac{\alpha+\lambda}{\lambda}\right)^k \geq 0.$$

Since for $k \geq \hat{n}$, $g'(k+0) \geq 0$, we deduce

$$\sum_{k=-[-s]}^{x} \frac{g'(k+0)}{\alpha+\lambda}\left(\frac{\alpha+\lambda}{\lambda}\right)^k \geq 0, \forall x \geq x_0.$$

It follows that $H_s'(x+0) \geq 0, \forall x \geq x_0$. Therefore

$$H_s(x) \leq H_s(y), \forall y \geq x \geq x_0.$$

Noting that $H_s(y) \leq \psi_s(y), \forall y \geq s$, we deduce trivially

$$H_s(x) \leq K + \psi_s(y), \forall y \geq x \geq x_0,$$

and the proof has been completed.

14.9. ERGODIC APPROACH FOR POISSON DEMAND WITH LEAD TIME

14.9.1. STUDY OF THE FUNCTION $\psi_s(x)$. We consider now the case $\alpha = 0$. We first get

$$g(x) = Ef(x - D(L)),$$

so

$$g(x) = \begin{cases} -px + \lambda pL, & \text{if } x \leq 0 \\ \quad -p\sum_{k=[x]+1}^{\infty}(x-k)\frac{(\lambda L)^k}{k!}\exp -\lambda L & \text{if } x \geq 0 \\ \qquad +h\sum_{k=0}^{[x]}(x-k)\frac{(\lambda L)^k}{k!}\exp -\lambda L, & \end{cases}$$

We next compute the function $\psi_s(x)$. We have for $x \geq s$

$$\psi_s(x) = \frac{1}{\lambda}\left\{\sum_{j=0}^{[x-s]}(g(x-j)-g(s)) + \exp -\lambda L \sum_{j=[x-s]+1}^{\infty}(g(x-j)-g(s))\sum_{k=j+1}^{\infty}\frac{(L\lambda)^k}{k!}\right\},$$

and for $x \leq s$

$$\psi_s(x) = \frac{1}{\lambda}\sum_{j=0}^{\infty}(g(x-j)-g(s))\sum_{k=j+1}^{\infty}\frac{(L\lambda)^k}{k!}.$$

In the sequel, to simplify a little we assume s integer. We can give explicit formulas for $\psi_s(x)$, depending on the sign of s. The computation is tedious but not difficult. It is sufficient to consider $x \geq s$.

Assume $s \leq x \leq 0$ integer we have

$$\psi_s(x) = pL(s-x) + \frac{1}{2}\lambda L^2 p - \frac{p}{\lambda}\exp -\lambda L \sum_{j=0}^{[x]-s}\frac{(\lambda L)^j}{j!} \cdot\left[\left(x - \frac{1}{2}[x] - \frac{1}{2}j\right)([x]-j+1) - s(x-j) + \frac{s^2}{2} - \frac{s}{2}\right]. \tag{14.9.1}$$

We use the fact that

$$g(x) - g(s) = p(s-x),$$

for $s < x < 0$.

The next case is $s < 0 < x$. Noting that, in that case,

$$g(x) - g(s) = p(s - x) + (h + p)E(x - D(L))^+,$$

it is easy to check that the formula giving $\psi_s(x)$ is the formula (14.9.1) with an additional term

$$\frac{h+p}{\lambda} \sum_{j=0}^{[x]} E(x - j - D(L))^+.$$

This term is equal to

$$\frac{h+p}{\lambda} \exp -\lambda L \sum_{j=0}^{[x]} \frac{(\lambda L)^j}{j!} \left(x - \frac{1}{2}[x] - \frac{1}{2}j\right) ([x] - j + 1).$$

Collecting results we obtain, for $s < 0 < x$

$$\begin{aligned}
(14.9.2) \qquad \psi_s(x) = &-pL(x - s) + \frac{1}{2}\lambda L^2 p + \\
&+\frac{h}{\lambda} \exp -\lambda L \sum_{j=0}^{[x]} \frac{(\lambda L)^j}{j!} \left(x - \frac{1}{2}[x] - \frac{1}{2}j\right) ([x] - j + 1) \\
&-\frac{p}{\lambda} \exp -\lambda L \sum_{j=[x]+1}^{[x]-s} \frac{(\lambda L)^j}{j!} \left(x - \frac{1}{2}[x] - \frac{1}{2}j\right) ([x] - j + 1) \\
&+\frac{p}{\lambda} \exp -\lambda L \sum_{j=0}^{[x]-s} \frac{(\lambda L)^j}{j!} \left((x - j)s + \frac{s}{2} - \frac{s^2}{2}\right).
\end{aligned}$$

Consider now the case $0 < s$. We recall that

$$g(x) - g(s) = p(s - x) + (h + p)E[(x - D(L))^+ - (s - D(L))^+].$$

Again we have to add to the expression (14.9.1) an additional term given by

$$\begin{aligned}
&\frac{h+p}{\lambda} \left[\sum_{j=0}^{[x]-s} E[(x - j - D(L))^+ - (s - D(L))^+] \right. \\
&\left. + \exp -\lambda L \sum_{j=[x]-s+1}^{\infty} E[(x - j - D(L))^+ - (s - D(L))^+] \sum_{k=j+1} \frac{(L\lambda)^k}{k!} \right]
\end{aligned}$$

We can express the additional term multiplying $\frac{h+p}{\lambda}$ as follows

$$\begin{aligned}
&\exp -\lambda L \sum_{j=0}^{[x]-s} \sum_{k=0}^{s-1} (x - j - s) \frac{(\lambda L)^k}{k!} + \exp -\lambda L \sum_{j=0}^{[x]-s} \sum_{k=s}^{[x]-j} (x - j - k) \frac{(\lambda L)^k}{k!} \\
&+\exp -2\lambda L \left[\sum_{j=[x]-s+1}^{[x]} \left(\sum_{m=0}^{[x]-j} (x - j - s) \frac{(\lambda L)^m}{m!} - \sum_{m=[x]-j+1}^{s} (s - m) \frac{(\lambda L)^m}{m!} \right) \right. \\
&\left. \cdot \sum_{k=j+1}^{\infty} \frac{(\lambda L)^k}{k!} \right] - \exp -2\lambda L \sum_{j=[x]+1}^{\infty} \sum_{m=0}^{s} (s - m) \frac{(\lambda L)^m}{m!} \sum_{k=j+1}^{\infty} \frac{(\lambda L)^k}{k!}
\end{aligned}$$

After transformation we obtain

$$\begin{aligned}
&\exp -\lambda L \sum_{k=0}^{s-1} \frac{(\lambda L)^k}{k!}([x]-s+1)\left(x-\frac{[x]}{2}-\frac{s}{2}\right)\\
&+\exp -\lambda L \sum_{k=s}^{[x]} \frac{(\lambda L)^k}{k!}([x]-k+1)\left(x-\frac{[x]}{2}-\frac{k}{2}\right)\\
&+\exp -2\lambda L \sum_{j=[x]-s+1}^{[x]} \sum_{m=0}^{[x]-j} \sum_{k=j+1}^{\infty} (x-j-s)\frac{(\lambda L)^m}{m!}\frac{(\lambda L)^k}{k!}\\
&-\exp -2\lambda L \sum_{j=[x]-s+1}^{[x]} \sum_{m=[x]-j+1}^{s} (s-m)\frac{(\lambda L)^m}{m!} \sum_{k=j+1}^{\infty} \frac{(\lambda L)^k}{k!}\\
&-\sum_{j=[x]+1}^{\infty} \sum_{m=0}^{s} (s-m)\frac{(\lambda L)^m}{m!} \sum_{k=j+1}^{\infty} \frac{(\lambda L)^k}{k!}.
\end{aligned}$$

14.9.2. LOCAL MINIMUM. We next compute the derivative $\psi_s'(x+0)$. We first have for $s<x<0$

$$\psi_s'(x+0) = -pL - \frac{p}{\lambda}\exp -\lambda L \sum_{j=0}^{[x]-s}([x]-j+1-s)\frac{(\lambda L)^j}{j!}. \tag{14.9.3}$$

We next consider the case $s<0<x$. We have

$$\begin{aligned}
\psi_s'(x+0) = {}&-pL - \frac{p}{\lambda}\exp -\lambda L \sum_{j=[x]+1}^{[x]-s}([x]-j+1-s)\frac{(\lambda L)^j}{j!}\\
&+\frac{\exp -\lambda L}{\lambda}\sum_{j=0}^{[x]}\frac{(\lambda L)^j}{j!}(h([x]-j+1)+ps).
\end{aligned} \tag{14.9.4}$$

Finally for $0<s<x$ we have

$$\begin{aligned}
\psi_s'(x+0) = {}&-pL - \frac{p}{\lambda}\exp -\lambda L \sum_{j=0}^{[x]-s}([x]-j+1-s)\frac{(\lambda L)^j}{j!}\\
&+\frac{h+p}{\lambda}\exp -\lambda L\left[\sum_{j=0}^{s-1}\frac{(\lambda L)^j}{j!}([x]-s+1)+\sum_{j=s}^{[x]}\frac{(\lambda L)^j}{j!}([x]-j+1)\right.\\
&\left.\cdot\exp -\lambda L \sum_{j=[x]+1}^{[x]+s}\sum_{l=j-s+1}^{\infty}\frac{(\lambda L)^l}{l!}\sum_{k=[x]+1}^{j} C_l^{j-k}\right].
\end{aligned} \tag{14.9.5}$$

We can check for $s<0<x$ the formula

$$\psi_s'(x+1+0)-\psi_s'(x+0) = \exp -\lambda L\left[\frac{h}{\lambda}\sum_{j=0}^{[x]+1}\frac{(\lambda L)^j}{j!} - \frac{p}{\lambda}\sum_{j=[x]+2}^{[x]+1-s}\frac{(\lambda L)^j}{j!}\right].$$

Define for $n \geq 0$ integer

$$\xi(n) = \exp -\lambda L \left[\frac{h}{\lambda} \sum_{j=0}^{n+1} \frac{(\lambda L)^j}{j!} - \frac{p}{\lambda} \sum_{j=n+2}^{n+1-s} \frac{(\lambda L)^j}{j!} \right].$$

We have

$$\xi(n+1) - \xi(n) = \frac{(\lambda L)^{n+2}}{(n+2)!} \frac{\exp -\lambda L}{\lambda} \left[h + p - p \frac{(\lambda L)^{-s}}{(n+3)\cdots(n+2-s)} \right].$$

Define $n_0 = n_0(s)$ to be the first positive integer such that

$$(n+3)\cdots(n+2-s) \geq \frac{p(L\lambda)^{-s}}{h+p}$$

Lemma 14.5. *For $s \leq 0$, there exists one and only one value S which is a local minimum. i.e. such that*

$$\psi_s'(S+0) \geq 0, \psi_s'(S-1+0) < 0.$$

The minimum S is the unique value satisfying this condition

PROOF. n_0 is the minimum of $\xi(n)$. We first check that

$$\xi(0) \geq 0 \Rightarrow \xi(n) \geq 0, \forall n. \tag{14.9.6}$$

Hence

$$\psi_s'(n+0) \text{ is increasing.}$$

If $\psi_s'(0+) \geq 0$, then $\psi_s'(n+0) \geq 0, \forall n$. Therefore $S = 0$. If $\psi_s'(0+) < 0$, noting that $\psi_s'(n+0) > 0$, for n sufficiently large, there exists one and only one S satisfying the property of local minimum.

Let us prove the property (14.9.6). Note that

$$\xi(0) \geq 0 \Rightarrow h(1+\lambda L) \geq p \sum_{j=2}^{1-s} \frac{(\lambda L)^j}{j!}.$$

Therefore

$$\xi(n) \geq \frac{p \exp -\lambda L}{\lambda(1+\lambda L)} \left[\sum_{j=2}^{1-s} \frac{(\lambda L)^j}{j!} \sum_{j=0}^{n+1} \frac{(\lambda L)^j}{j!} - \sum_{j=n+2}^{n+1-s} \frac{(\lambda L)^j}{j!} (1+\lambda L) \right].$$

Let us set

$$b_j = \frac{(\lambda L)^j}{j!}, \quad \nu = 1 - s \geq 1.$$

The positivity of $\xi(n)$ will be a consequence of the inequality

$$\sum_{j=0}^{\nu} b_j \sum_{k=0}^{n+1} b_k - \sum_{j=0}^{n+\nu} b_j \sum_{k=0}^{1} b_k$$

To prove the inequality it is enough to prove that

$$\frac{\sum_{j=0}^{n+\nu} b_j}{\sum_{j=0}^{n+1} b_k} \text{ is decreasing in } n.$$

Comparing the terms in n and in $n+1$ this amounts to proving

$$b_{n+1+\nu}\sum_{k=0}^{n+1} b_k \le b_{n+2}\sum_{j=0}^{n+\nu} b_j,$$

and it is sufficient to prove

$$b_{n+1+\nu}\sum_{k=0}^{n+1} b_k \le b_{n+2}\sum_{j=\nu-1}^{n+\nu} b_j = b_{n+2}\sum_{k=0}^{n+1} b_{\nu-1+k}$$

It is thus sufficient to check that

$$\frac{b_k}{b_{n+2}} \le \frac{b_{\nu-1+k}}{b_{n+1+\nu}}, \forall k \le n+1.$$

This results in checking the property

$$(k+1)\cdots(n+2) \le (k+\nu)\cdots(n+1+\nu), \forall k \le n+1,$$

which is satisfied.

It remains to consider the case $\xi(0) < 0$, in which case $\xi(n_0) < 0$.

Since $\xi(n) > 0$ for n sufficiently large, there exists $S_1 > n_0$ such that

$$\xi(S_1) \ge 0, \xi(S_1 - 1) < 0.$$

Then

$$\xi(n) \ge 0 \ \forall n \ge S_1; \ \xi(n) < 0 \ \forall 0 \le n < S_1.$$

Therefore S_1 is uniquely defined.

Hence

$$\psi_s'(n+1+0) - \psi_s'(n+0) \ge 0, \forall n \ge S_1,$$

$$\psi_s'(n+1+0) - \psi_s'(n+0) < 0, \forall 0 \le n < S_1.$$

Therefore S_1 is the minimum of $\psi_s'(n+0)$. If this minimum is positive, we can conclude that $S = 0$. If not, it is easily checked that there exists a unique local minimum $S > S_1$ This completes the proof. □

We now turn to the case $s > 0$, in which case naturally $S(s) \ge s > 0$. We first compute the derivative

$$\psi_s'(x+0) = R_s([x]) = -pL + \frac{h}{\lambda}\exp -\lambda L \sum_{j=0}^{[x]-s} \frac{(\lambda L)^j}{j!}([x]-s-j+1)$$

$$+\frac{h+p}{\lambda}\exp -\lambda L \left[\sum_{k=0}^{s-1} \frac{(\lambda L)^k}{k!}([x]-s+1)\right.$$

$$+\sum_{k=s}^{[x]} \frac{(\lambda L)^k}{k!}([x]-k+1) - \sum_{k=0}^{[x]-s} \frac{(\lambda L)^k}{k!}([x]-k+1-s)$$

$$\left.+\exp -\lambda L \sum_{j=[x]+1-s}^{[x]} \sum_{m=0}^{[x]-j} \sum_{k=j+1}^{\infty} \frac{(\lambda L)^m}{m!}\frac{(\lambda L)^k}{k!}\right].$$

The function $R_s(n), n \geq s$ is increasing in n. We can indeed check

$$R_s(n+1) - R_s(n) = \frac{h}{\lambda} \exp -\lambda L \sum_{j=0}^{n+1-s} \frac{(\lambda L)^j}{j!}$$

$$+\frac{h+p}{\lambda} \exp -\lambda L \left[\sum_{j=n+2-s}^{n+1} \frac{(\lambda L)^j}{j!} - \exp -\lambda L \sum_{m=0}^{s-1} \sum_{j=n+2-s}^{n+2-m} \frac{(\lambda L)^j}{j!} \frac{(\lambda L)^m}{m!} \right],$$

and

$$R_s(n+1) - R_s(n) \geq 0, \forall n \geq s$$

We next consider $R_s(s), s \geq 0$. We first check

$$R_s(s) = -pL + \frac{h}{\lambda} \exp -\lambda L + \frac{h+p}{\lambda} \exp -\lambda L \Big[- \sum_{j=s+1}^{\infty} \frac{(\lambda L)^j}{j!}$$

$$+ \exp -\lambda L \sum_{j=1}^{s} \sum_{m=0}^{s-j} \sum_{k=j+1}^{\infty} \frac{(\lambda L)^m}{m!} \frac{(\lambda L)^k}{k!} \Big].$$

It is easy to check that $R_s(s)$ is an increasing function. We can proceed to define $S(s)$ in a unique way. We shall need $\hat{s}$ to be the first positive integer such that $R_s(s) \geq 0$. This number is uniquely defined.

Lemma 14.6. *For $s \geq \hat{s}$ we have $S(s) = s$ and for $0 \leq s \leq \hat{s}$ $S(s)$ is the first integer such that $R_s(S) \geq 0$.*

PROOF. The proof is immediate from the monotonicity property of $R_s(x)$. □

LITTLE'S LAW FOR THE LEAD TIME CASE:. We consider Little's law for positive lead time L. Suppose that at starting time the initial stock is x. Define successively

$$\theta = \inf\{t | x - D(t) \leq 0\};$$
$$\tau = \inf\{t | x - D(t) \leq s\}.$$

Note that

$$D(\tau) - D(\theta) = -s, \text{ if } \tau \neq \theta, \theta, \tau \neq 0.$$

At time τ an order is made of size $S - s$ if $\tau > 0$, and of size $S - x$ if $\tau = 0$. It is delivered at time $\tau + L$. So after delivery the inventory is $S - D(\tau + L) + D(\tau)$. Note that we assume s integer (positive or negative).

The possibility of backlog exists only when $\theta < \tau + L$. Otherwise there is no backlog and no waiting time. If $S - D(\tau + L) + D(\tau) > 0$ then the backlog is absorbed at time $\tau + L$. If it is not positive, the population $D(\tau + L) - D(\tau) - S$ has to wait till the next delivery. Defining

$$\tau_1 = \inf\{t \geq \tau + L | S - D(t) + D(\tau) \leq s\} - \tau - L,$$

then this part of the population will be served at time $\tau + L + \tau_1 + L$, Note that

$$S - D(\tau + L) + D(\tau) < s \Rightarrow \tau_1 = 0.$$

Moreover

$$S - D(\tau + L) + D(\tau) > s \Rightarrow S - D(\tau + L + \tau_1) + D(\tau) = s.$$

Exercise 14.12. Check the formula

$$\lambda E \tau_1 = E(S - s - D(\tau + L) + D(\tau))^+.$$

A cycle is the interval $\min(\theta, \tau), \tau + L$. We consider the demands which arrive during a cycle, the backlog they constitute till they are served and the corresponding waiting time. If $\theta > \tau + L$ there is no backlog and no waiting time. The population which is served at time $\tau + L$ is

$$-s + \min(S, D(\tau + L) - D(\tau)) \text{ if } \tau > 0,$$

and

$$-x + \min(S, D(L)) \text{ if } \tau = 0.$$

If the inventory at $\tau + L$ is negative then all demands during the cycle cannot be satisfied at time $\tau + L$. The leftover will be served at time $\tau + L + \tau_1 + L$. Its level is $(S - D(\tau + L) + D(\tau))^-$. The average backlog can thus be expressed as

$$B = \int_0^1 (D(\theta + \rho(\tau + L - \theta)^+ - D(\theta))d\rho + (S - D(\tau + L) + D(\tau))^- \frac{L}{\tau + L - \theta}$$
$$+ (S - D(\tau + L) + D(\tau))^- \tau_1 \mathbb{1}_{\{S - s - D(\tau+L) + D(\tau) > 0\}}$$

Then the expected backlog is

$$EB = \frac{\lambda}{2} E(\tau + L - \theta)^+ + LE \frac{(S - D(\tau + L) + D(\tau))^-}{\tau + L - \theta}$$
$$+ E\left[\tau_1 \mathbb{1}_{\{S - s - D(\tau+L) + D(\tau) > 0\}} \frac{(S - D(\tau + L) + D(\tau))^-}{\tau + L - \theta}\right].$$

We can consider now the waiting time. First we note that the whole population has to wait during the cycle an average time $\frac{1}{2}(\tau + L - \theta)^+$. In addition the carry over has to wait an additional time $L + \tau_1$, where τ_1 can be 0. Weighting with the population ratios we obtain an expected average waiting time

$$EWT = \frac{1}{2} E(\tau + L - \theta)^+ + LE \frac{(S - D(\tau + L) + D(\tau))^-}{D(\tau + L) - D(\theta)}$$
$$+ E\left[\tau_1 \mathbb{1}_{\{S - s - D(\tau+L) + D(\tau) > 0\}} \frac{(S - D(\tau + L) + D(\tau))^-}{D(\tau + L) - D(\theta)}\right]$$

Little's Law:

$$EB = \lambda EWT,$$

is not verified. It is verified approximately if we neglect the probability of carry over. Namely we assume that

$$\text{Prob } \{D(\tau + L) - D(\tau) > S\} \sim 0,$$

which means

$$\sum_{j=S+1}^{\infty} \frac{(\lambda L)^j}{j!} \sim 0.$$

Another approximation consists in replacing the total population $D(\tau + L) - D(\theta)$ by $\lambda(\tau + L - \theta)$ in the coefficient of $(S - D(\tau + L) + D(\tau))^-$. These two random variables have the same expected value. With these approximations Little's Law remains valid. These approximations require a not too large lead time.

14.10. POISSON DEMAND WITH LEAD TIME: USE OF INVENTORY POSITION

14.10.1. FORMULATION OF THE PROBLEM. Considering the problem defined in the preceding section, we change slightly the notation, since the state of the system will not be the inventory, but a new concept, called *Inventory Position*. The impulse control will still be denoted by

$$M(t;V) = \sum_{\{n|\theta_n<t\}} v_n,$$

and this time there is no constraint on the stopping times θ_n The inventory will be denoted by $\tilde{y}_x(t;V)$. However, x does not represent the initial inventory, but the initial inventory position.

With respect to the real inventory, the inventory position adds all the orders which have not been yet delivered. So, the evolution of the inventory position $y_x(t;V)$ is described by

$$y_x(t;V) = x - D(t) + M(t;V),$$

where x is its initial value, which will become a part of the inventory at time L. Clearly we have the relation

$$\tilde{y}_x(t+L;V) = y_x(t;V) - D(t+L) + D(t).$$

This relation defines the inventory at times larger than L. Before L the inventory is not influenced by decisions taken in the lead time interval. If we do not have a record of orders before 0, then the inventory before L is what it is at time 0 depleted by the demand. At any rate the corresponding cost is fixed and there is no optimization possibility.

The cost corresponding to an impulse control V is given by

$$J_x(V) = E\left[\sum_{n=0}^{\infty}(K+cv_n)\exp -\alpha\theta_n + \int_L^{\infty} f(\tilde{y}_x(t;V))\exp -\alpha t dt\right].$$

In this writing we have not included the cost on the interval $0, L$ which is a constant. This cost can be immediately written as

$$J_x(V) = E\Bigg[\sum_{n=0}^{\infty}(K+cv_n)\exp -\alpha\theta_n + \exp -\alpha L\int_0^{\infty} f(y_x(t;V) - D(t+L) + D(t))\exp -\alpha t dt\Bigg]$$

We note that $D(t+L) - D(t)$ is independent of $y_x(t;V)$. Therefore, we can introduce the function

$$\phi(x) = \exp -\alpha L \; Ef(x - D(L)),$$

and the cost is given by

$$J_x(V) = E\left[\sum_{n=0}^{\infty}(K+cv_n)\exp -\alpha\theta_n + \int_0^{\infty} \phi(y_x(t;V))\exp -\alpha t dt\right].$$

We obtain for the inventory position the same problem as that for the inventory without lead time but with an integral cost $\phi(x)$ instead of $f(x)$. The corresponding function $g(x)$ is the same as the one entering in the problem considered in the previous section, see equation (14.8.7).

14.10.2. PROBLEM FOR THE VALUE FUNCTION. The value function

$$u(x) = \inf_V J_x(V),$$

satisfies the set of inequalities and complementarity conditions

$$\begin{gathered} u(x) + cx \leq K + \inf_{\eta \geq x}(c\eta + u(\eta)); \\ (\alpha + \lambda)u(x) - \lambda u(x-1) \leq \phi(x); \\ [(\alpha + \lambda)u(x) - \lambda u(x-1) - \phi(x)][u(x) + cx - K - \inf_{\eta \geq x}(c\eta + u(\eta))] = 0. \end{gathered}$$

14.10.3. s, S **POLICY.** As we have done in the case without delays, we look for a function $u_s(x)$ such that

$$\begin{gathered} u_s(x) + cx = u_s(s) + cs, \forall x \leq s \\ (\alpha + \lambda)u_s(x) - \lambda u_s(x-1) = \phi(x), \forall x \geq s \\ u_s(s) + cs = K + \inf_{\eta \geq s}(c\eta + u_s(\eta)). \end{gathered}$$

The last expression is an equation for s, once the function $u_s(x)$ is uniquely determined. To define $u_s(x)$ for any s, we need the value of $u_s(s)$ This number is given by

$$u_s(s) = \frac{\phi(s) + \lambda c}{\alpha}.$$

This motivates the introduction of the function $H_s(x)$ solution of

$$\begin{gathered} H_s(x) = 0, \forall x \leq s \\ (\alpha + \lambda)H_s(x) - \lambda H_s(x-1) = g(x) - g(s), \forall x \geq s \\ 0 = K + \inf_{\eta \geq s} H_s(\eta), \end{gathered}$$

and the function $g(x)$ is defined by

$$g(x) = \phi(x) + \rho cx,$$

which is the function already introduced in the last section (14.8.7). We have the relation

$$H_s(x) = u_s(x) + cx - cs - \frac{\lambda c + \phi(s)}{\alpha}.$$

We recall the the formula for $H_s(x)$

$$H_s(x) = \sum_{j=0}^{[x-s]} \frac{(g(x-j) - g(s))\lambda^j}{(\alpha + \lambda)^{j+1}}, \forall x \geq s; \quad H_s(x) = 0, \forall x \leq s. \tag{14.10.1}$$

The theory developed in section 14.6 applies, with a minimum of $g(x)$ which is $\hat{n}$ which is a positive integer instead of 0.

Consider first $S(s)$ which minimizes $H_s(\eta), \eta \geq s$. We have

$$S(s) = s, \text{ if } s \geq \hat{n}$$

Consider next the case $s < \hat{n}$. As in section 14.6 we prove that $H_s(x)$ reaches its minimum on (s, ∞). Picking the smallest minimum $S(s)$ we define the function $S(s)$ for $s < \hat{n}$. Necessarily

$$S(s) \geq \hat{n}.$$

There remains to find s. It must satisfy the equation in s

$$\min_{\eta \geq s} H_s(\eta) = -K.$$

Again the theory of section 14.6 proves the existence and uniqueness of the solution s. We next can state the equivalent of Theorem 14.2

Theorem 14.4. *We assume $c\alpha - p \exp -\alpha L < 0$. Then, the s, S policy is optimal.*

14.11. ERGODIC THEORY FOR LEAD TIME WITH INVENTORY POSITION

14.11.1. FORMULATION OF THE PROBLEM.

We consider now $\alpha = 0$. The problem reduces to the following. First

$$g(x) = Ef(x - D(L)),$$

and $\phi(x) = g(x)$.

The function $H_s(x)$ satisfies

$$H_s(x) - H_s(x-1) = \frac{g(x) - g(s)}{\lambda}, \forall x \geq s; \quad H_s(x) = 0, \forall x \leq s.$$

The pair s, S are defined by

$$H_s(S) = \inf_{\eta \geq s} H_s(\eta) = -K.$$

We get

$$H_s(x) = \frac{1}{\lambda} \sum_{j=0}^{[x-s]} (g(x-j) - g(s)).$$

The derivative to the right satisfies

$$H_s'(x+0) = \frac{1}{\lambda} \sum_{j=0}^{[x-s]} g'(x-j+0).$$

The function $H_s(x)$ being piece-wise linear the computation of $H_s(x)$ reduces to the case x integer with the formula

$$H_s(x) = H_s([x]) + H_s'([x]+0)(x - [x]).$$

So we can consider x integer. Then

$$H_s(x) = \frac{1}{\lambda} \sum_{j=0}^{x+[-s]} (g(x-j) - g(s)) = \frac{1}{\lambda} \sum_{k=-[-s]}^{x} (g(k) - g(s)),$$

and

$$H_s'(x+0) = \frac{1}{\lambda} \sum_{k=-[-s]}^{x} g'(k+0).$$

To obtain s, S, S integer we proceed as follows: we write the conditions

$$H_s'(S+0) = \frac{1}{\lambda} \sum_{k=-[-s]}^{S} g'(k+0) \geq 0;$$

$$H_s'(S-1+0) = \frac{1}{\lambda} \sum_{k=-[-s]}^{S-1} g'(k+0) < 0,$$

and

$$H_s(S) = \frac{1}{\lambda} \sum_{k=-[-s]}^{S} (g(k) - g(s)) = -K.$$

Next we use the identity

$$g(k+1) = g(-[-s]) + \sum_{j=-[-s]}^{k} g'(j+0).$$

The second relation writes

$$-K\lambda = (g(-[-s]) - g(s))(S + [-s] + 1) + \sum_{j=-[-s]}^{S-1} g'(j+0)(S-j).$$

Using the first conditions we can write

$$\begin{aligned} &\sum_{j=-[-s]}^{S-1} g'(j+0)j < K\lambda + (g(-[-s]) - g(s))(S + [-s] + 1); \\ &\sum_{j=-[-s]}^{S} g'(j+0)j \geq K\lambda + (g(-[-s]) - g(s))(S + [-s] + 1). \end{aligned} \tag{14.11.1}$$

To which we add

$$\begin{aligned} &\sum_{j=-[-s]}^{S-1} g'(j+0) < 0; \\ &\sum_{j=-[-s]}^{S} g'(j+0) \geq 0. \end{aligned} \tag{14.11.2}$$

14.11.2. APPROXIMATION. If we approximate s by its integer value, we get the following system

$$\begin{aligned} &\sum_{j=s}^{S-1} g'(j+0)j < K\lambda; \\ &\sum_{j=s}^{S} g'(j+0)j \geq K\lambda. \end{aligned} \tag{14.11.3}$$

$$\begin{aligned} &\sum_{j=s}^{S-1} g'(j+0) < 0; \\ &\sum_{j=s}^{S} g'(j+0) \geq 0. \end{aligned} \tag{14.11.4}$$

First we notice that if we neglect K, we get a base stock policy

$$s = S = \hat{n}.$$

Next we approximate the conditions by the equations

$$\sum_{j=s}^{S} g'(j+0) = 0,$$

and

$$\sum_{j=s}^{S} jg'(j+0) = K\lambda.$$

We proceed with continuous approximation of these relations

$$\begin{aligned} &g(s) = g(S) \\ &\int_s^S xg'(x+0)dx = K\lambda \end{aligned} \tag{14.11.5}$$

14.11.3. INTERPRETATION. We can recover the system (14.11.5) by performing a direct reasoning on the cost function. We consider a pair s, S which we assume integers. Applying an s, S policy the evolution of the stock is described as follows. Set $Q = S - s$ and define successively the sequence of stopping times

$$T_{n+1} = T_n + \tau_n, \;\; T_0 = 0,$$

with

$$\tau_n = \inf\{t > 0 | D(T_n + t) - D(T_n) \geq Q\},$$

and set $\tau_0 = \tau$.

The stopping times τ_n are independent random variables and identically distributed as the random variable τ. Consider the evolution of the stock starting with S, denoted $y_S(t)$. We have

$$y_S(t) = S - D(t) + D(T_n), \; T_n \leq t < T_{n+1},$$

and we can interpret T_n as

$$T_{n+1} = \inf\{t \geq T_n | y_S(t-0) \leq s\},$$

and

$$y_S(T_n) = S, \;\; y_S(T_n - 0) \leq s.$$

Define

$$C_\alpha(s, Q) = E\int_0^\infty \exp -\alpha t \phi(y_S(t))dt.$$

We compute

$$C_\alpha(s, Q) = \sum_{n=0}^\infty E\int_{T_n}^{T_{n+1}} \exp -\alpha t \phi(S - D(t) + D(T_n))dt,$$

$$C_\alpha(s, Q) = \sum_{n=0}^\infty E \exp -\alpha T_n \int_0^{\tau_n} \exp -\alpha t \phi(S - D(T_n + t) + D(T_n))dt,$$

and by independence considerations

$$C_\alpha(s, Q) = \sum_{n=0}^\infty E(\exp -\alpha\tau)^n E\int_0^\tau \exp -\alpha t \phi(S - D(t))dt.$$

Finally we obtain

$$C_\alpha(s, Q) = \frac{E\int_0^\tau \exp -\alpha t \phi(S - D(t))dt}{1 - E(\exp -\alpha\tau)} \tag{14.11.6}$$

Consider now the evolution of the stock starting at x, any value. We define first

$$\tau_x = \inf\{t > 0 | D(t) \geq x - s\}.$$

We define next successively $T_{n,x}$ as follows:

$$\begin{array}{l} T_{0,x} = \tau_x; \;\; T_{n+1,x} = T_{n,x} + \tau_{n,x} \\ \tau_{n,x} = \inf\{t > 0 | D(T_{n,x} + t) - D(T_{n,x}) \geq Q\} \end{array}.$$

We define $y_x(t)$ by the relations

$$\begin{array}{l} y_x(t) = x - D(t), t < \tau_x \\ y_x(t) = S - D(t) + +D(T_{n,x}), \; T_{n,x} \leq t < T_{n+1,x} \\ y_x(T_{n,x}) = S; \; x_x(T_{n,x} - 0) \leq s \end{array}$$

We consider

$$u_\alpha(x) = E \int_0^\infty \exp -\alpha t \phi(y_x(t)) dt.$$

In the above notation, we do not write explicitly s, S. On the other hand we note explicitly α, since we will let α tend to 0. Note also that

$$u_\alpha(S) = C_\alpha(s, Q) = C_\alpha.$$

In the same way we have obtained the formula for C_α, we can check the following formula

$$u_\alpha(x) = E \int_0^{\tau_x} \exp -\alpha t \phi(x - D(t)) dt + C_\alpha E \exp -\alpha \tau_x$$

From the preceding formula we notice that

$$x \leq s \Rightarrow \tau_x = 0 \Rightarrow u_\alpha(x) = C_\alpha.$$

A key point now is to obtain the probability distribution of τ_x. To simplify a little we will assume x, s integers. We have

$$P(\tau_x \leq t) = P(D(t) \geq x - s) = \exp -\lambda t \sum_{j=x-s}^{\infty} \frac{(\lambda t)^j}{j!}.$$

This probability distribution has a density given by

$$\frac{d}{dt} P(\tau_x \leq t) = \lambda \exp -\lambda t \frac{(\lambda t)^{x-s-1}}{(x-s-1)!}.$$

This has sense whenever $x - s \geq 1$ If $x = s$ then

$$\frac{d}{dt} P(\tau_x \leq t) = \lambda \exp -\lambda t,$$

and the density is 0 for $x < s$.

It is then possible to compute the function $E \exp -\alpha \tau_x$. We leave as an exercise to show that

$$E \exp -\alpha \tau_x = \left(\frac{\lambda}{\alpha + \lambda} \right)^{x-s}.$$

Next we can write

$$E \int_0^{\tau_x} \exp -\alpha t \phi(x - D(t)) dt = E \int_0^\infty \exp -\alpha t 1\!\!1_{\{\tau_x > t\}} \phi(x - D(t)) dt$$

$$E \int_0^{\tau_x} \exp -\alpha t \phi(x - D(t)) dt = E \int_0^\infty \exp -\alpha \rho t 1\!\!1_{\{D(t) < x - s\}} \phi(x - D(t)) dt.$$

Introducing the notation

$$\tilde{\phi}_s(x) = \phi(x) 1\!\!1_{x > s},$$

we have

$$E\int_0^{\tau_x} \exp -\alpha t \phi(x - D(t))dt = E\int_0^{\infty} \exp -\rho t \tilde{\phi}_s(x - D(t))dt$$

Collecting results, we obtain the formula

$$u_\alpha(x) = E\int_0^{\infty} \exp -\alpha t \tilde{\phi}_s(x - D(t))dt + C_\alpha \left(\frac{\lambda}{\alpha+\lambda}\right)^{x-s}.$$

In ergodic theory, one is interested in the behavior of $\alpha u_\alpha(x)$ as $\alpha \to 0$, for any fixed x. From the above formula, it is clear that

$$\lim_{\alpha\to 0} \alpha u_\alpha(x) = \lim_{\alpha\to 0} \alpha C_\alpha.$$

It is easy to check that

$$\lim_{\alpha\to 0} \alpha C_\alpha = \frac{\lambda H_s(S)}{Q},$$

where

$$H_s(S) = E\int_0^{\infty} \tilde{\phi}_s(S - D(t))dt.$$

It is easy to check the relation

$$H_s(S) - H_s(S-1) = \frac{\tilde{\phi}_s(S)}{\lambda} = \frac{\phi(S)}{\lambda}.$$

By induction, one checks the formula

$$H_s(S) = \sum_{j=0}^{Q-1} \frac{\phi(S-j)}{\lambda}.$$

Finally we have obtained

$$\lim_{\alpha\to 0} \alpha u_\alpha(x) = \lim_{\alpha\to 0} \alpha C_\alpha = \sum_{k=s+1}^{S} \frac{\phi(k)}{Q}.$$

Recalling that, from standard ergodic theory arguments

$$\lim_{\alpha\to 0} \alpha u_\alpha(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(y_x(t))dt = \sum_{k=s+1}^{S} \frac{\phi(k)}{Q}.$$

The right hand side can be interpreted as $E\phi(X)$ where X is a random variable distributed uniformly on $s+1, \cdots, S$.

We can now proceed with the optimization of the parameters s, S. We optimize the ergodic cost

$$\lim_{\alpha\to 0} \alpha E\left[(K + cQ)\sum_0^{\infty} \exp -\alpha T_n + \int_0^{\infty} \exp -\alpha t \phi(y_x(t))dt\right].$$

From previous results, this amounts to

$$L(s, Q) = \frac{\lambda K}{Q} + \sum_{k=1}^{Q} \frac{g(s+k)}{Q},$$

recalling

$$g(x) = Ef(x - D(L)).$$

To write necessary conditions we consider the continuous approximation which writes

$$L(s, Q) = \frac{\lambda K}{Q} + \frac{1}{Q}\int_0^Q g(s + x)dx.$$

It is easy to recover the conditions (14.11.5).

CHAPTER 15

INVENTORY CONTROL WITH DIFFUSION DEMAND

15.1. INTRODUCTION

We continue the study of Inventory Control in continuous time. We want to consider a quite general demand process, combining deterministic rate and Poisson jumps as in Chapter 14 plus a diffusion term. We follow the work of [**9**] and the complement [**4**]. We will use the full power of Q.V.I (Quasi-Variational Inequalities) to solve the Dynamic Programming problem associated to the impulse control problem. We then check that the optimal feedback is obtained by an s, S policy.

15.2. PROBLEM FORMULATION

15.2.1. DEMAND PROCESS. The demand process is composed of three parts: a deterministic part with constant rate ν. It was called λ in

Chapter 14, but λ will be used for the compound Poisson part. We consider a probability space $\Omega, \mathcal{A}, P$ on which are defined a Poisson process $n(t)$ with intensity rate λ, a Wiener process $w(t)$ and a sequence of independent identically distributed random variables $\xi_1, \cdots, \xi_i, \cdots$ All these random variables and stochastic processes are mutually independent. The random variables ξ_i have a probability density μ on R^+. We define the compound Poisson process by

$$N(t) = \sum_{i \leq n(t)} \xi_i.$$

The demand on an interval $(0, t)$ is then given by

$$D(t) = \nu t + N(t) + \sigma w(t),$$

where σ is a positive coefficient. Let $\mathcal{F}^t = \sigma(D(s), s \leq t)$

An impulse control is a sequence

$$\theta_n, v_n,$$

where θ_n is a stopping time with respect to the filtration $\mathcal{F}^t$ and v_n is a random variable $\mathcal{F}^{\theta_n}$ measurable. Denoting by V an impulse control, the corresponding inventory is described by the formula

$$y_x(t; V) = x - D(t) + M(t; V),$$

with

$$M(t) = M(t; V) = \sum_{\{n | \theta_n < t\}} v_n$$

15.2.2. CONTROL PROBLEM. Let again

$$f(x) = hx^+ + px^-.$$

We define the cost functional

$$J_x(V) = E\left[\sum_{n=0}^{\infty}(K + cv_n)\exp -\alpha\theta_n + \int_0^\infty f(y_x(t;V))\exp -\alpha t dt\right], \tag{15.2.1}$$

and the value function

$$u(x) = \inf\ J_x(V)$$

To obtain the relations of Dynamic Programming we can assume sufficient smoothness and write the optimality principle. If we do not order for a small amount of time δ then we can write

$$u(x) \leq \delta f(x) + (1 - \alpha\delta)Eu(x - D(\delta)),$$

and

$$Eu(x - D(\delta)) = Eu(x - \delta\nu - N(\delta) - \sigma w(\delta)).$$

Formally

$$N(\delta) = \begin{cases} 0, & \text{w.p. } 1 - \lambda\delta \\ \in (\xi, \xi + d\xi), & \text{w.p. } \lambda\delta\mu(\xi)d\xi \end{cases}$$

and $w(\delta)$ is Gaussian with mean 0 and variance δ. Therefore

$$Eu(x - D(\delta)) \simeq (1 - \lambda\delta)Eu(x - \nu\delta - \sigma w(\delta)) + \lambda\delta\int_0^\infty u(x - \xi)\mu(\xi)d\xi$$

Expanding up to the first order in δ assuming second order smoothness of u we obtain the inequality

$$Au(x) + \alpha u(x) \leq f(x),$$

in which A is the integro-differential operator

$$Au(x) = -\frac{1}{2}\sigma^2 u''(x) + \nu u'(x) - \lambda\int_0^\infty (u(x - \xi) - u(x))\mu(\xi)d\xi. \tag{15.2.2}$$

In case we make an order of size v, then the inventory becomes instantaneously $x + v$ and we we write the inequality

$$u(x) \leq K + cv + u(x + v),\ \forall v \geq 0$$

so

$$u(x) \leq M(u)(x),$$

with

$$M(u)(x) = K + \inf_{v>0}[cv + u(x + v)] \tag{15.2.3}$$

and since one of the two decisions must be taken, we have the complementarity slackness condition

$$(Au(x) + \alpha u(x) - f(x))(u(x) - M(u)(x)) = 0.$$

15.2.3. QUASI-VARIATIONAL INEQUALITY. We shall assume

$$\exists \bar{\pi} > 0, \text{ such that } \int_0^\infty \exp \bar{\pi}\xi \; \mu(\xi) d\xi \quad = \quad +\infty$$

(15.2.4)
$$\int_0^\infty \exp \pi\xi \; \mu(\xi) d\xi < +\infty, \; \forall \pi < \bar{\pi}$$

We look for a function $u(x)$ which is $C^1, a.e.$ twice differentiable with linear growth such that

(15.2.5)
$$\begin{array}{cc} Au(x) + \alpha u(x) \leq f(x) & \text{a.e.} \\ u(x) \leq M(u)(x) & \\ (Au(x) + \alpha u(x) - f(x))(u(x) - M(u)(x)) = 0 & \text{a.e.} \end{array}$$

We call this system of relations a Q.V.I. (Quasi-Variational Inequality)

15.3. s, S POLICY

15.3.1. NOTATION. Let us set

$$\begin{array}{rcl} G(x) & = & u(x) + cx; \\ g(x) & = & f(x) + \alpha cx, \end{array}$$

then the Q.V.I is slightly modified into

(15.3.1)
$$\begin{array}{rcl} AG(x) + \alpha G(x) & \leq & g(x) + c\nu + c\lambda\bar{\xi} \\ G(x) & \leq & K + \inf_{\eta > x} G(\eta) \end{array}$$

(15.3.2)
$$(AG(x) + \alpha G(x) - (g(x) + c\nu + c\lambda\bar{\xi})) \left(G(x) - K - \inf_{\eta > x} G(\eta) \right) = 0,$$

in which $\bar{\xi}$ denotes the mean of a random variable with probability μ

$$\bar{\xi} = \int_0^{+\infty} \xi\mu(\xi) d\xi.$$

If s is a real number we define $G_s(x)$ by solving the following problem

(15.3.3)
$$\begin{array}{rcll} AG_s(x) + \alpha G_s(x) & = & g(x) + c\nu + c\lambda\bar{\xi} & x > s \\ G_s(x) & = & K + \inf_{\eta > s} G_s(\eta) & x \leq s \end{array}$$

Since we want the solution to be C^1 we must impose the condition

(15.3.4)
$$G_s'(s) = 0$$

15.3.2. FUNCTION $H_s(x)$. It is natural to look for the derivative

$$H_s(x) = G_s'(x)$$

which is the solution of

(15.3.5)
$$\begin{array}{rcll} AH_s(x) + \alpha H_s(x) & = & g'(x) & x > s \\ H_s(x) & = & 0 & x \leq s \end{array}$$

Since $g'(x)$ is bounded we look for bounded solutions

Proposition 15.1. *We assume* (15.2.4)*. Then, for any s there exists a unique continuous bounded function solution* (15.3.5)*. The solution is C^2 on $(s, +\infty)$.*

PROOF. Let us prove uniqueness. It means that a bounded solution of

$$\begin{aligned} AH_s(x) + \alpha H_s(x) &= 0 \quad x > s \\ H_s(x) &= 0 \quad x \leq s \end{aligned}$$

is 0. This follows from Maximum principle considerations. We have

$$AH_s(x)+\alpha H_s(x) = -\frac{1}{2}\sigma^2 H_s"(x)+\nu H_s'(x)+(\lambda+\alpha)H_s(x)-\lambda\int_0^{x-s} H_s(x-\xi)\mu(\xi)d\xi.$$

Suppose $H_s(x)$ becomes strictly positive. Let us prove that it cannot have a local maximum. If it has a local maximum $x^* > s$ then we may assume that it is the smallest one. In such a point

$$H_s'(x^*) = 0, \quad H_s"(x^*) < 0,$$

and

$$\int_0^{x^*-s} H_s(x^* - \xi)\mu(\xi)d\xi \leq H_s(x^*)\int_0^{x^*-s} \mu(\xi)d\xi \leq H_s(x^*).$$

Therefore $AH_s(x^*) > 0$, which is a contradiction. Therefore $H_s'(x) > 0$. Since $H_s(x)$ is bounded, we must have $H_s'(x) \to 0$ as $x \to +\infty$. Therefore there is a point x^* such that $H_s"(x^*) < 0$. In such a point, necessarily $AH_s(x^*) > 0$, and we have a contradiction again. Therefore $H_s(x) \leq 0$ and a similar reasoning shows that it cannot become strictly negative.

To prove existence, we shall construct an explicit solution. However, we need preliminaries. Let us denote

$$\hat{\mu}(\pi) = \int_0^{+\infty} \exp \pi\xi \; \mu(\xi)d\xi, \quad \pi < \bar{\pi}$$

We consider the function

$$\chi(\pi) = -\frac{\sigma^2}{2}\pi^2 - \nu\pi + \lambda + \alpha - \lambda\hat{\mu}(\pi),$$

for $\pi \leq \bar{\pi}$. We first note that

$$\begin{aligned} \chi(\bar{\pi}) &= \chi(-\infty) = -\infty; \\ \chi'(\pi) &= -\sigma^2\pi - \nu - \lambda\int_0^{+\infty} \xi \exp \pi\xi \; \mu(\xi)d\xi; \\ \chi"(\pi) &= -\sigma^2 - \lambda\int_0^{+\infty} \xi^2 \exp \pi\xi \; \mu(\xi)d\xi < 0; \\ \chi'(\bar{\pi}) &= -\infty, \;\; \chi'(-\infty) = +\infty, \;\; \chi'(0) < 0. \end{aligned}$$

Therefore $\chi'(\pi)$ is strictly decreasing from $+\infty$ to -∞. There exists a unique $\pi_0 < 0$ such that $\chi'(\pi_0) = 0$. Hence π_0 is the maximum of $\chi(\pi)$. Since $\chi(0) = \alpha$, we get $\chi(\pi_0) > 0$. Therefore $\chi(\pi)$ has only two zeros β_1 and β_2 with

$$\beta_2 < \pi_0 < 0 < \beta_1 < \bar{\pi}.$$

Let also

$$\chi_0(\pi) = -\frac{\sigma^2}{2}\pi^2 - \nu\pi + \lambda + \alpha,$$

and let β_0 be the positive root of $\chi_0(\pi) = 0$. Since $\chi(\pi) < \chi_0(\pi)$, we have $\chi(\beta_0) < 0$ and thus

$$\beta_1 < \beta_0 < \bar{\pi}.$$

To proceed we introduce the following problem:

$$-\frac{1}{2}\sigma^2\Gamma''(x) + \nu\Gamma'(x) + (\lambda+\alpha)\Gamma(x) - \lambda\int_0^x \Gamma(x-\xi)\mu(\xi)d\xi = 0, \quad \forall x > 0$$

$$\Gamma(0) = 1, \quad \Gamma(+\infty) = 0 \tag{15.3.6}$$

We shall prove in Lemma 15.1 below that there exists one and only one solution of (15.3.6) such that

$$\exp -\beta_0 x < \Gamma(x) < \exp -\beta_1 x \tag{15.3.7}$$

We will also prove the property

$$\Gamma'(0) = \frac{2\nu}{\sigma^2} + \beta_2. \tag{15.3.8}$$

We can now exhibit the solution of (15.3.5), namely

$$H_s(x) = \int_s^x \Gamma(x-\xi)Q(\xi)d\xi, \tag{15.3.9}$$

with

$$Q(x) = \frac{2}{\sigma^2}\int_x^{+\infty} \exp \beta_2(\eta - x)\, g'(\eta)d\eta. \tag{15.3.10}$$

From the estimate (15.3.7) we check that $H_s(x)$ remains bounded and a direct calculation shows that it is indeed a solution of equation (15.3.5). □

Let us state the

Lemma 15.1. *There exists one and only one solution of equation* (15.3.6) *such that* (15.3.7) *holds.*

Proof. The uniqueness is proved by maximum principle considerations as for $H_s(x)$ in Proposition 15.1. To prove the existence, we first check that

$$\underline{\Gamma}(x) = \exp -\beta_0 x \qquad \overline{\Gamma}(x) = \exp -\beta_1 x,$$

satisfy

$$-\frac{1}{2}\sigma^2\overline{\Gamma}''(x) + \nu\overline{\Gamma}'(x) + (\lambda+\alpha)\overline{\Gamma}(x) - \lambda\int_0^x \overline{\Gamma}(x-\xi)\mu(\xi)d\xi \geq 0; \tag{15.3.11}$$

$$-\frac{1}{2}\sigma^2\underline{\Gamma}''(x) + \nu\underline{\Gamma}'(x) + (\lambda+\alpha)\underline{\Gamma}(x) - \lambda\int_0^x \underline{\Gamma}(x-\xi)\mu(\xi)d\xi \leq 0. \tag{15.3.12}$$

We then consider the sequence

$$-\frac{1}{2}\sigma^2\Gamma_{n+1}''(x) + \nu\Gamma_{n+1}'(x) + (\lambda+\alpha)\Gamma_{n+1}(x) = \lambda\int_0^x \Gamma_n(x-\xi)\mu(\xi)d\xi;$$

$$\Gamma_0(x) = \underline{\Gamma}(x),$$

with boundary conditions

$$\Gamma_n(0) = 1, \quad \Gamma_n(+\infty) = 0.$$

Suppose that

$$\underline{\Gamma}(x) \leq \Gamma_n(x) \leq \overline{\Gamma}(x),$$

then $\Gamma_{n+1}(x)$ is easily obtained by solving a 2nd order differential equation, the important property is to check that

$$\Gamma_n(x) \leq \Gamma_{n+1}(x) \leq \overline{\Gamma}(x).$$

Let us prove it for $n = 0$. This amounts to

$$\underline{\Gamma}(x) \leq \Gamma_1(x) \leq \overline{\Gamma}(x),$$

which is easy to check, thanks to (15.3.11), (15.3.12) and the equation of $\Gamma_1(x)$. Assuming the property holds at n, one checks immediately that it holds at $n+1$. Therefore the sequence $\Gamma_n(x)$ is a monotone increasing sequence. The limit $\Gamma(x)$ is a solution of (15.3.6) and satisfies (15.3.7) par construction. Let us check (15.3.8). We first check that $\Gamma'(+\infty) = 0$. Indeed, we write equation (15.3.6) as

$$-\frac{1}{2}\sigma^2\Gamma"(x) + \nu\Gamma'(x) + (\lambda+\alpha)\Gamma(x) = \theta(x), \tag{15.3.13}$$

with

$$\theta(x) = \lambda\int_0^x \Gamma(x-\xi)\mu(\xi)d\xi,$$

and we can state the estimate

$$0 \leq \theta(x) \leq \lambda\hat{\mu}(\beta_1)\exp -\beta_1 x.$$

Now we can obtain the solution of (15.3.13) as a second-order differential equation with right hand-side. Noting

$$\bar{\beta}_0 = -\beta_0 - \frac{2\nu}{\sigma^2},$$

which is the negative root of $\chi_0(\pi) = 0$, we obtain easily

$$\Gamma(x) = C\exp -\beta_0 x + \frac{2}{\sigma^2(\beta_0 - \bar{\beta}_0)}$$
$$\left[\int_0^x \exp -\beta_0(x-\xi)\theta(\xi)d\xi + \int_x^{+\infty}\exp\bar{\beta}_0(\xi - x)\theta(\xi)d\xi\right],$$

and

$$\Gamma'(x) = -\beta_0 C\exp -\beta_0 x - \frac{2}{\sigma^2(\beta_0 - \bar{\beta}_0)}$$
$$\cdot\left[\beta_0\int_0^x \exp -\beta_0(x-\xi)\theta(\xi)d\xi + \bar{\beta}_0\int_x^{+\infty}\exp\bar{\beta}_0(\xi - x)\theta(\xi)d\xi\right].$$

But

$$0 \leq \int_0^x \exp -\beta_0(x-\xi)\theta(\xi)d\xi \leq \frac{\lambda\hat{\mu}(\beta_1)}{\beta_0 - \beta_1}(\exp -\beta_1 x - \exp -\beta_0 x)$$
$$0 \leq \int_x^{+\infty}\exp\bar{\beta}_0(\xi - x)\theta(\xi)d\xi \leq \frac{\lambda\hat{\mu}(\beta_1)}{\beta_1 - \bar{\beta}_0}\exp -\beta_1 x$$

which implies $\Gamma'(+\infty) = 0$. We can then test equation (15.3.6) with $\exp\pi x$, with $\pi < 0$. We set

$$\hat{\Gamma}(\pi) = \int_0^{+\infty}\exp\pi x\,\Gamma(x)dx,$$

which is well defined.

Using boundary conditions, we compute easily

$$\frac{\sigma^2}{2}(\Gamma'(0) - \pi) - \nu + \hat{\Gamma}(\pi)\chi(\pi) = 0,$$

and applying this relation with $\pi = \beta_2$ we obtain the relation (15.3.8).

We must have $\Gamma'(0) < 0$. Let us check indeed this property.

Let us prove the property

$$\frac{2\nu}{\sigma^2} + \beta_2 < -\beta_1 \tag{15.3.14}$$

which will imply $\Gamma'(0) < 0$. This follows from noting first $\chi'(-\frac{\nu}{\sigma^2}) < 0$. Therefore

$$-\frac{\nu}{\sigma^2} > \pi_0 > \beta_2 \Rightarrow -2\frac{\nu}{\sigma^2} - \beta_2 > \beta_2.$$

Next

$$\chi\left(-2\frac{\nu}{\sigma^2} - \beta_2\right) = \lambda\hat{\mu}(\beta_2) - \lambda\hat{\mu}\left(-2\frac{\nu}{\sigma^2} - \beta_2\right) < 0.$$

Combining results, the inequality (15.3.14) follows immediately. □

15.3.3. OBTAINING THE s, S POLICY. We find now $s, G_s(x)$ solution of (15.3.3), (15.3.4). We first construct for any s, $G_s(x)$ solution of the first equation (15.3.3) and of (15.3.4). For s fixed we simply define

$$\begin{aligned} G_s(x) &= G_s(s) + \int_s^x H_s(\xi)d\xi, \quad x > s \\ &= G_s(s), \quad x \le s \end{aligned} \tag{15.3.15}$$

We define in this way a C^1 function, and (15.3.4) is clearly satisfied. Let us check that the first equation (15.3.3) is satisfied for a convenient choice of $G_s(s)$. The function is also C^2 for $x > s$. We reinterpret the first equation (15.3.3) using the definition of $G_s(x)$. We obtain, after rearrangements, for $x > s$

$$-\frac{\sigma^2}{2}H_s'(x) + \nu H_s(x) + \alpha G_s(s) + (\lambda + \alpha)\int_s^x H_s(\xi)d\xi$$
$$-\lambda\int_0^{x-s}\mu(\xi)\left(\int_s^{x-\xi} H_s(\eta)d\eta\right)d\xi = g(x) + c\nu + c\lambda\bar{\xi}.$$

On the other hand, from (15.3.5), by integration between s and x, we get

$$-\frac{\sigma^2}{2}H_s'(x) + \frac{\sigma^2}{2}H_s'(s) + \nu H_s(x) + (\lambda + \alpha)\int_s^x H_s(\xi)d\xi -$$
$$\lambda\int_0^{x-s}\mu(\xi)\left(\int_s^{x-\xi} H_s(\eta)d\eta\right)d\xi = g(x) - g(s).$$

Noting that $H_s'(s) = Q(s)$ and comparing the two relations we obtain the value of $G_s(s)$, namely

$$G_s(s) = \frac{\frac{\sigma^2}{2}Q(s) + g(s) + c\nu + c\lambda\bar{\xi}}{\alpha}. \tag{15.3.16}$$

To proceed we make the standard assumption

$$p - \alpha c > 0. \tag{15.3.17}$$

We can give an explicit formula for $Q(x)$, namely

$$Q(x) = \frac{2}{\beta_2\sigma^2}(p - \alpha c - (h + p)\exp\beta_2 x^-), \tag{15.3.18}$$

therefore we have

$$Q'(x) = \frac{2(h+p)}{\sigma^2}\exp -\beta_2 x \;\mathbb{1}_{x<0},$$

hence $Q(x)$ is an increasing function with

$$Q(-\infty) = \frac{2}{\beta_2\sigma^2}(p - \alpha c), \quad Q(x) = -\frac{2}{\beta_2\sigma^2}(h + \alpha c), \ \forall x \geq 0$$

Proposition 15.2. *Assuming* (15.3.17)*, the function $G_s(x)$ defined by* (15.3.15) *attains its infimum for $x > s$. Taking the smallest minimum, we define in a unique way a function $S(s)$ such that*

$$G_s(S(s)) = \inf_{\eta \geq s} G_s(\eta). \tag{15.3.19}$$

Proof. Let $x_0 < 0$ the unique point such that $Q(x_0) = 0$. It is given explicitly by

$$\exp -\beta_2 x_0 = \frac{p - \alpha c}{h + p}.$$

Then for $s > x_0$ we have $H_s(x) > 0$. Therefore $G_s(x)$ is increasing for $x > s$, and the minimum is attained in s. So in this case, $S(s) = s$. We next assume that $s < x_0$. For $s < x < x_0$ we have $H_s(x) < 0$ and thus $G_s(x)$ decreases on (s, x_0). Considering $x > 0$ we can write

$$H_s(x) = \int_s^0 \Gamma(x - \xi)Q(\xi)d\xi = -\frac{2}{\sigma^2\beta_2}(h + \alpha c)\int_0^x \Gamma(\xi)d\xi.$$

Using the estimates on $\Gamma(x)$ and the minimum value of $Q(x)$ we can show easily the inequality

$$H_s(x) \geq \frac{2}{\sigma^2\beta_2}(p - \alpha c)\frac{1 - \exp\beta_1 s}{\beta_1}\exp -\beta_1 x - \frac{2}{\sigma^2\beta_2}(h + \alpha c)\frac{1 - \exp -\beta_0 x}{\beta_0}, \ \forall x \geq 0.$$

Therefore also

$$\liminf_{x\to+\infty} H_s(x) \geq -\frac{2}{\sigma^2\beta_2\beta_0}(h + \alpha c),$$

which implies also $G_s(x) \to +\infty$ as $x \to +\infty$. Collecting results we see that $G_s(x)$ attains its infimum for $x > s$. Taking the smallest infimum we define in a unique way $S(s)$. Necessarily $S(s) > x_0$. □

Consider $s < x_0$, $S(s)$ minimizes $G_s(x)$, hence

$$G_s'(S(s)) = H_s(S(s)) = 0; \ G"_s(S(s)) = H_s'(S(s)) > 0.$$

Also

$$S'(s)H_s'(S(s)) + \frac{\partial H_s}{\partial s}(S(s)) = 0.$$

Since

$$\frac{\partial H_s}{\partial s}(S(s)) = -\Gamma(S(s) - s)Q(s) > 0,$$

we obtain

$$\begin{matrix} S'(s) < 0, & \text{if } s < x_0 \\ S'(s) = 1, & \text{if } s > x_0 \end{matrix}. \tag{15.3.20}$$

Let us check that

$$S'(x_0 - 0) = -1. \tag{15.3.21}$$

Indeed, since $s < x_0 < S(s)$ we can write

$$\int_s^{x_0} \Gamma(S(s) - \xi)Q(\xi)d\xi + \int_{x_0}^{S(s)} \Gamma(S(s) - \xi)Q(\xi)d\xi = 0.$$

As $s \uparrow x_0$ we have $S(s) \downarrow x_0$. In the two integrals

$$\Gamma(S(s) - \xi) \simeq 1, \quad Q(\xi) \simeq Q'(x_0)(\xi - x_0),$$

which implies

$$(s - x_0)^2 \simeq (S(s) - x_0)^2,$$

and taking account of the signs we get

$$x_0 - s \simeq S(s) - x_0,$$

hence

$$\frac{S(s) - x_0}{s - x_0} \to -1,$$

which implies the result.

We can turn to the problem of finding s such that the second equation (15.3.3) is satisfied. Since we look for a single number, we need one equation. We write

$$G_s(s) = K + \inf_{\eta > s} G_s(\eta),$$

which is equivalent of

$$0 = K + \int_s^{S(s)} H_s(\eta) d\eta. \tag{15.3.22}$$

It is then natural to study the function

$$\gamma(s) = \int_s^{S(s)} H_s(\eta) d\eta,$$

we have $\gamma(s) = 0$ for $s \geq x_0$ and for $s < x_0$

$$\gamma'(s) = \int_s^{S(s)} \frac{\partial H_s}{\partial s}(\eta) d\eta = -Q(s) \int_s^{S(s)} \Gamma(\eta - s) d\eta > 0.$$

Note also that $\gamma(s)$ is C^1. Remembering the estimate on $\Gamma(x)$ we obtain the estimate

$$\gamma'(s) \geq -\frac{Q(s)}{\beta_0}(1 - \exp -\beta_0(S(s) - s)) \geq -\frac{Q(s)}{\beta_0}(1 - \exp -\beta_0(x_0 - s)).$$

Let us take $s < s_1 < x_0$, since $Q(s)$ increases we get easily

$$\gamma'(s) \geq -\frac{Q(s_1)}{\beta_0}(1 - \exp -\beta_0(x_0 - s_1)) > 0.$$

This implies that $\gamma(s) \to -\infty$ as $s \to -\infty$. Therefore is strictly monotone increasing from $-\infty$ to 0, as s increases from $-\infty$ to x_0. It follows that there exists a unique $s < x_0$ such that $\gamma(s) = -K$. We can now state the

Theorem 15.1. *We assume* (15.2.4) (15.3.17). *Then there exists a unique pair* $s, G_s(x)$ *such that* $G_s(x) \in C^1$ *and* (15.3.3), (15.3.4) *are satisfied.*

Proof. We have proved that the number s such that $\gamma(s) = -K$ and the corresponding function $G_s(x)$ defined by (15.3.15), (15.3.16) satisfy the (15.3.3), (15.3.4). It is unique since its derivative must coincide with $H_s(x)$ which is uniquely defined by (15.3.5), and the value $G_s(s)$ is uniquely defined by (15.3.16). Furthermore, because of the second relation (15.3.3), the number s must satisfy $\gamma(s) = -K$, which has a unique solution. □

Setting $S(s) = S$ for the unique value s defined in Theorem 15.1, we obtain the s, S policy.

15.4. SOLVING THE Q.V.I

It remains to show that the s, S policy solves the problem. This amounts to showing that the function $G_s(x)$ defined by (15.3.3), (15.3.4) solves the Q.V.I. (15.3.1), (15.3.2).

Theorem 15.2. *Same assumptions as in Theorem 15.1. The function $G_s(x)$ defined by* (15.3.3), (15.3.4) *solves the Q.V.I.* (15.3.1), (15.3.2).

PROOF. We first note that

$$\inf_{\eta>x} G_s(\eta) = \inf_{\eta>s} G_s(\eta),\ \forall x \leq s.$$

Indeed we note that for $x < \eta < s$ we have

$$G_s(\eta) = G_s(s) > \inf_{\eta>s} G_s(\eta).$$

Therefore the function $G_s(x)$ satisfies the complementarity slackness condition (15.3.2). It remains to show that it satisfies the inequalities (15.3.1). This amounts to

$$AG_s(x) + \alpha G_s(x) \leq g(x) + c\nu + c\lambda\bar{\xi} \quad x \leq s; \tag{15.4.1}$$

$$G_s(x) \leq K + \inf_{\eta>x} G_s(\eta) \quad x \geq s. \tag{15.4.2}$$

We begin with (15.4.1). Since $G_s(x) = G_s(s)$ for $x \leq s$, the inequality (15.4.1) amounts to

$$\alpha G_s(s) \leq g(x) + c\nu + c\lambda\bar{\xi} \quad x \leq s,$$

and from the definition of $G_s(s)$ we get

$$\frac{\sigma^2}{2}Q(s) + g(s) \leq g(x) \quad x \leq s,$$

which is true since $Q(s) \leq 0$ and $g(s) \leq g(x) \quad x \leq s \leq 0$.

We turn now to (15.4.2). Define

$$B_s(x) = G_s(x) - \inf_{\eta>x} G_s(\eta).$$

We want to show that

$$B_s(x) \leq K,\ \forall x \geq s. \tag{15.4.3}$$

We begin with $s \leq x \leq x_0$. The function $G_s(x)$ decreases on (s, x_0), therefore

$$\begin{aligned} G_s(x) &\leq G_s(s) = K + \inf_{\eta>s} G_s(\eta) \\ &\leq K + \inf_{\eta>x} G_s(\eta), \end{aligned}$$

and (15.4.3) is satisfied. We can then assume $x > x_0$. Suppose there exists a point $x' > x_0$ such that $B_s(x') \geq K$. We can assume it is the first point such that this occurs. Necessarily

$$B_s(x') = K, \quad B_s(x) < K, \forall x, s < x < x'$$

Consider $G_s(\eta),\ \eta > x'$. Its infimum is attained, since $G_s(x) \to +\infty$, as $x \to +\infty$. Taking the smallest minimum we define a unique point x_3 such that

$$G_s(x_3) = \inf_{\eta \geq x'} G_s(\eta). \tag{15.4.4}$$

We can claim that $x_3 \neq x'$ since

$$B_s(x') = G_s(x') - G_s(x_3) = K,$$

therefore x_3 is a local minimum, which implies

$$G_s'(x_3) = 0, \quad G"_s(x_3) > 0 \tag{15.4.5}$$

Define next x_2 to be the smallest maximum of $G_s(\eta),\ \eta \leq x_3$

$$G_s(x_2) = \sup_{\eta \leq x_3} G_s(\eta) \tag{15.4.6}$$

The maximum is attained since

$$\sup_{\eta \leq x_3} G_s(\eta) = \sup_{s \leq \eta \leq x_3} G_s(\eta),$$

and the function $G_s(x)$ is continuous. Let us check that $x_2 \geq x'$. Indeed suppose that $x_2 < x'$. Then

$$\inf_{\eta \geq x_2} G_s(\eta) \leq \inf_{\eta \geq x'} G_s(\eta),$$

and

$$G_s(x_2) - \inf_{\eta \geq x_2} G_s(\eta) \geq G_s(x') - \inf_{\eta \geq x'} G_s(\eta) = K$$

which contradicts the definition of x'. Moreover $x_2 \neq x_3$ since

$$G_s(x_2) - G_s(x_3) \geq G_s(x') - G_s(x_3) = K.$$

Therefore x_2 is also a local maximum and we have

$$G_s'(x_2) = 0 \quad G"_s(x_2) < 0. \tag{15.4.7}$$

So we have $x' \leq x_2 < x_3$. Let us prove the property

$$G_s(x) - G_s(y) \leq G_s(x_2) - G_s(x_3),\ \forall x \leq y \leq x_3. \tag{15.4.8}$$

Indeed, if $x < x'$ then

$$G_s(x) - G_s(y) \leq G_s(x) - \inf_{\eta \geq x} G_s(\eta) < K \leq G_s(x_2) - G_s(x_3),$$

and if $x \geq x'$ then $x' \leq x \leq y \leq x_3$ and

$$G_s(x) - G_s(y) \leq G_s(x_2) - G_s(x_3),$$

from the definition of x_2 and x_3.

Finally, let us define x_1 such that

$$G_s(x_1) = \inf_{\eta \leq x'} G_s(\eta). \tag{15.4.9}$$

For the same reasons as in the definition of x_2 the point x_1 is well defined as the smallest minimum. Moreover $x_1 \neq x'$. Indeed

$$G_s(x') = K + \inf_{\eta \geq x'} G_s(\eta),$$

and recalling that $x' > x_0 > s$ and $G_s(x)$ decreases on (s, x_0)

$$G_s(x') \geq K + \inf_{\eta \geq s} G_s(\eta) = G_s(s) > G_s(x_0) \geq \inf_{\eta \leq x'} G_s(\eta) = G_s(x_1),$$

consequently x_1 is a local minimum and thus

$$G_s'(x_1) = 0, \quad G"_s(x_1) > 0. \tag{15.4.10}$$

Note the sequence

$$s < x_0 \leq x_1 < x' \leq x_2 < x_3. \tag{15.4.11}$$

We can now proceed with a contradiction argument, to show that a point like x' cannot exist. We recall that for $x > s$ the function $G_s(x)$ satisfies

$$-\frac{1}{2}\sigma^2 G_s''(x) + \nu G_s'(x) + (\alpha+\lambda)G_s(x) - \lambda\int_0^\infty G_s(x-\xi)\mu(\xi)d\xi = l(x), \tag{15.4.12}$$

with

$$l(x) = g(x) + c\nu + c\lambda\bar{\xi}.$$

Recalling that $x_0 < 0$, we examine two possibilities:

- If $x_2 < 0$ then we write (15.4.12) at points x_1 and x_2. We have

$$-\frac{1}{2}\sigma^2 G_s''(x_1) + \nu G_s'(x_1) + (\alpha+\lambda)G_s(x_1) - \lambda\int_0^\infty G_s(x_1-\xi)\mu(\xi)d\xi = l(x_1);$$

$$-\frac{1}{2}\sigma^2 G_s''(x_2) + \nu G_s'(x_2) + (\alpha+\lambda)G_s(x_2) - \lambda\int_0^\infty G_s(x_2-\xi)\mu(\xi)d\xi = l(x_2),$$

and we use

$$G_s'(x_1) = G_s'(x_2) = 0;$$

$$G_s''(x_1) > 0, \quad G_s''(x_2) < 0;$$

$$G_s(x_2) - G_s(x_2-\xi) > 0, \quad G_s(x_1) - G_s(x_1-\xi) < 0;$$

$$l(x_1) > l(x_2),$$

therefore necessarily $G_s(x_1) > G_s(x_2)$ which is impossible.

- If $x_2 \geq 0$ then we write (15.4.12) at points x_2 and x_3. We have

$$-\frac{1}{2}\sigma^2 G_s''(x_3) + \nu G_s'(x_3) + (\alpha+\lambda)G_s(x_3) - \lambda\int_0^\infty G_s(x_3-\xi)\mu(\xi)d\xi = l(x_3)$$

We have then

$$G_s'(x_3) = G_s'(x_2) = 0$$

$$G_s''(x_3) > 0 \quad G_s''(x_2) < 0$$

$$G_s(x_2-\xi) - G_s(x_3-\xi) \leq G_s(x_2) - G_s(x_3)$$

$$l(x_2) < l(x_3),$$

it follows that $G_s(x_2) - G_s(x_3) < 0$ which is also a contradiction. Hence a point like x' cannot exist, which proves that (15.4.3) is satisfied and completes the proof that the function $G_s(x)$ solves the Q.V.I. □

15.5. ERGODIC THEORY

15.5.1. NOTATION AND MODEL. We want now let $\alpha \to 0$. We will index the functions with α. So we have the quasi-variational inequality

$$\begin{aligned} &AG_\alpha(x) + \alpha G_\alpha(x) \leq g_\alpha(x) \\ &G_\alpha(x) \leq K + \inf_{\eta > x} G_\alpha(\eta) \end{aligned} \tag{15.5.1}$$

$$(AG_\alpha(x) + \alpha G_\alpha(x) - g_\alpha(x))\left(G_\alpha(x) - K + \inf_{\eta > x} G_\alpha(\eta)\right) = 0,$$

with

$$g_\alpha(x) = f(x) + \alpha c x + c\nu + c\lambda\bar{\xi}. \tag{15.5.2}$$

The function $G_\alpha(x)$ is obtained from an s,S policy. We define successively $\Gamma_\alpha(x)$ by

$$\begin{aligned} & -\frac{1}{2}\sigma^2\Gamma_\alpha''(x)+\nu\Gamma'_\alpha(x)+\lambda\Gamma_\alpha(x)-\lambda\int_0^x\Gamma_\alpha(x-\xi)\mu(\xi)d\xi \\ & \qquad +\alpha\Gamma_\alpha(x)=0, \quad \forall x>0 \\ & \Gamma_\alpha(0)=1, \quad \Gamma_\alpha(+\infty)=0 \end{aligned} \tag{15.5.3}$$

Let for $\pi<\bar{\pi}$, $\chi_\alpha(\pi)$ be the function

$$\chi_\alpha(\pi)=-\frac{\sigma^2}{2}\pi^2-\nu\pi+\lambda-\lambda\hat{\mu}(\pi)+\alpha \tag{15.5.4}$$

The maximum is attained in a point $\pi_0<0$ which does not depend on α. We know that $\chi_\alpha(\pi)$ has two zeros $\beta_{2\alpha}<\pi_0$ and $0<\beta_{1\alpha}<\bar{\pi}$. We next define

$$Q_\alpha(x)=\frac{2}{\sigma^2}\int_x^{+\infty}\exp\beta_{2\alpha}(\eta-x)g'_\alpha(\eta)d\eta \tag{15.5.5}$$

and for any s and $x\geq s$

$$H_{\alpha,s}(x)=\int_s^x\Gamma_\alpha(x-\xi)Q_\alpha(\xi)d\xi \tag{15.5.6}$$

the function $H_{\alpha,s}(x)$ is extended by 0 for $x<s$. We then define the function $G_{\alpha,s}(x)$ as follows

$$G'_{\alpha,s}(x)=H_{\alpha,s}(x) \quad G_{\alpha,s}(s)=\frac{\frac{\sigma^2}{2}Q_\alpha(s)+g_\alpha(s)}{\alpha} \tag{15.5.7}$$

We also need

$$\chi_{0\alpha}(\pi)=-\frac{\sigma^2}{2}\pi^2-\nu\pi+\lambda+\alpha \tag{15.5.8}$$

which has two zeros, $\beta_{0\alpha}>0$ and $\bar{\beta}_{0\alpha}<0$.

We know that

$$\exp-\beta_{0\alpha}x\leq\Gamma_\alpha(x)\leq\exp-\beta_{1\alpha}x$$

Let $x_{0\alpha}$ be the unique zero of $Q_\alpha(x)$. We know that there exists a unique $s_\alpha<x_{0\alpha}$ such that

$$0=K+\inf_{x\geq s_\alpha}\int_{s_\alpha}^x H_{\alpha,s_\alpha}(\xi)d\xi. \tag{15.5.9}$$

The function

$$G_\alpha(x)=G_{\alpha,s_\alpha}(x) \tag{15.5.10}$$

is the solution of the Q.V.I (15.5.1). We also define the number $S_\alpha>x_{0\alpha}$ that

$$0=K+\int_{s_\alpha}^{S_\alpha}H_{\alpha,s_\alpha}(\xi)d\xi \quad H_{\alpha,s_\alpha}(S_\alpha)=0 \tag{15.5.11}$$

15.5.2. PASSING TO THE LIMIT. We can now let $\alpha \to 0$. We set

$$\chi(\pi) = -\frac{\sigma^2}{2}\pi^2 - \nu\pi + \lambda - \lambda\hat{\mu}(\pi) = \lim_{\alpha\to 0}\chi_\alpha(\pi) \tag{15.5.12}$$

$$\chi_0(\pi) = -\frac{\sigma^2}{2}\pi^2 - \nu\pi + \lambda = \lim_{\alpha\to 0}\chi_{0\alpha}(\pi) \tag{15.5.13}$$

The roots converge

$$\beta_{0\alpha} \to \beta_0 > 0 \quad \bar{\beta}_{0\alpha} \to \bar{\beta}_0 < 0$$
$$\beta_{2\alpha} \to \beta_2 < 0 \quad \beta_{1\alpha} \to \beta_1 = 0.$$

The second property $\beta_1 = 0$ instead of a strictly positive number has consequences on the limit of $\Gamma_\alpha(x)$. The property that $\Gamma_\alpha(x)$ tends to 0 as $x \to +\infty$ cannot be maintained, so the limit problem for $\Gamma_\alpha(x)$ is the following

$$-\frac{1}{2}\sigma^2\Gamma"(x) + \nu\Gamma'(x) + \lambda\Gamma(x) - \lambda\int_0^x \Gamma(x-\xi)\mu(\xi)d\xi \quad = \quad 0 \tag{15.5.14}$$

$$\exp -\beta_0 x \le \Gamma(x) \le 1, \quad \Gamma(0) = 1, \quad \Gamma'(x) \text{ bounded,}$$

and

$$\Gamma_\alpha(x) \uparrow \Gamma(x), \quad \text{as } \alpha \downarrow 0.$$

Define

$$\theta(x) = \lambda\int_0^x \Gamma(x-\xi)\mu(\xi)d\xi,$$

then we can write the formula

$$\Gamma(x) = C\exp -\beta_0 x + \frac{2}{\sigma^2(\beta_0 - \bar{\beta}_0)}$$
$$\left[\int_0^x \exp -\beta_0(x-\xi)\theta(\xi)d\xi + \int_x^{+\infty} \exp \bar{\beta}_0(\xi - x)\theta(\xi)d\xi\right],$$

with

$$1 = C + \frac{2}{\sigma^2(\beta_0 - \bar{\beta}_0)}\int_0^{+\infty} \exp \bar{\beta}_0\xi \ \theta(\xi)d\xi.$$

Also setting

$$\theta_\alpha(x) = -\alpha\Gamma_\alpha(x) + \lambda\int_0^x \Gamma_\alpha(x-\xi)\mu(\xi)d\xi,$$

we can also write

$$\Gamma_\alpha(x) = C_\alpha \exp -\beta_0 x + \frac{2}{\sigma^2(\beta_0 - \bar{\beta}_0)}$$
$$\cdot\left[\int_0^x \exp -\beta_0(x-\xi)\theta_\alpha(\xi)d\xi + \int_x^{+\infty} \exp \bar{\beta}_0(\xi - x)\theta_\alpha(\xi)d\xi\right],$$

with

$$1 = C_\alpha + \frac{2}{\sigma^2(\beta_0 - \bar{\beta}_0)}\int_0^{+\infty} \exp \bar{\beta}_0\xi \ \theta_\alpha(\xi)d\xi.$$

From these formulas and the monotone convergence of $\Gamma_\alpha(x)$ to $\Gamma(x)$ it is easy to check that $\Gamma_\alpha(x)$ converges uniformly to $\Gamma(x)$ on any compact of R^+. Also

$$Q_\alpha(x) \to Q(x) \text{ uniformly,}$$

with

$$\begin{aligned} Q(x) &= \frac{2}{\sigma^2}\int_x^{+\infty} \exp\beta_2(\eta - x)f'(\eta)d\eta \\ &= \frac{2}{\beta_2\sigma^2}(p-(h+p)\exp\beta_2 x^-) \end{aligned}$$

We call

$$x_0 = \lim_{\alpha\to 0} x_{0\alpha} \quad \exp -\beta_2 x_0 = \frac{p}{h+p}.$$

For any s we have

$$H_{\alpha,s}(x) \to H_s(x),$$

uniformly on compact sets of R, with

(15.5.15) $$H_s(x) = \int_s^x \Gamma(x-\xi)Q(\xi)d\xi.$$

Consider now the function

$$\gamma_\alpha(s) = \inf_{x\geq s}\int_s^x H_{\alpha,s}(\xi)d\xi = \int_s^{S_\alpha(s)} H_{\alpha,s}(\xi)d\xi.$$

We first check that

(15.5.16) $$0 > s > \bar{s} \Rightarrow \limsup_{\alpha\to 0} S_\alpha(s) \leq B(\bar{s}),$$

for some bound $B(\bar{s}) > 0$ to be made precise below.

Without loss of generality we may assume $S_\alpha(s) > 0$. We have then $H_{\alpha,s}(S_\alpha(s)) = 0$. Therefore

$$-\int_s^{x_{0\alpha}} \Gamma_\alpha(S_\alpha(s)-\xi)Q_\alpha(\xi)d\xi = \int_{x_{0\alpha}}^{S_\alpha(s)} \Gamma_\alpha(S_\alpha(s)-\xi)Q_\alpha(\xi)d\xi,$$

hence

$$\int_0^{S_\alpha(s)} \Gamma_\alpha(S_\alpha(s)-\xi)Q_\alpha(\xi)d\xi \leq -\int_s^{x_{0\alpha}} \Gamma_\alpha(S_\alpha(s)-\xi)Q_\alpha(\xi)d\xi.$$

Using

$$Q_\alpha(\xi) = -\frac{2}{\beta_{2\alpha}\sigma^2}(h+\alpha c),\ \xi \geq 0\,; Q_\alpha(\xi) \geq \frac{2}{\beta_{2\alpha}\sigma^2}(p-\alpha c),$$

we deduce

$$\begin{aligned} (h+\alpha c)\int_0^{S_\alpha(s)} \Gamma_\alpha(\xi)d\xi &\leq (p-\alpha c)\int_s^{x_{0\alpha}} \Gamma_\alpha(S_\alpha(s)-\xi)d\xi \\ &\leq (p-\alpha c)(x_{0\alpha}-\bar{s}) \end{aligned}$$

Let $M > 0$ and set $S_\alpha^M(s) = \min(M, S_\alpha(s))$. We have

$$(h+\alpha c)\int_0^{S_\alpha^M(s)} \Gamma_\alpha(\xi)d\xi \leq (p-\alpha c)(x_{0\alpha}-\bar{s}).$$

Let $S^M(s) = \limsup_{\alpha\to 0} S_\alpha^M(s)$. By taking the limsup in the preceding inequality we obtain

$$h\int_0^{S^M(s)} \Gamma(\xi)d\xi \leq p(x_0 - \bar{s}).$$

However we have the property

$$\int_0^{+\infty} \Gamma(\xi)d\xi = +\infty. \tag{15.5.17}$$

Indeed if we consider for $\pi < 0$

$$\hat{\Gamma}(\pi) = \int_0^{+\infty} \exp \pi\xi \, \Gamma(\xi)d\xi,$$

we deduce from the equation (15.5.14) that

$$\hat{\Gamma}(\pi) = \frac{\sigma^2}{2}\frac{\pi - \beta_2}{\chi(\pi)}.$$

As $\pi \uparrow 0$, we know that $\chi(\pi) \to 0$, hence $\hat{\Gamma}(\pi) \uparrow +\infty$, which implies (15.5.17). We can then define a unique $B(\bar{s})$ such that

$$\int_0^{B(\bar{s})} \Gamma(\xi)d\xi = \frac{p(x_0 - \bar{s})}{h},$$

and thus

$$S^M(s) \leq B(\bar{s}),$$

and letting $M \uparrow +\infty$, we get the property (15.5.16).

Let us take $s < x_0$. We can assume also for α small $s < x_{0\alpha}$. Therefore $H_{\alpha,s}(S_\alpha(s)) = 0$. From the estimate (15.5.16) we can assume that for α small $S_\alpha(s)$ remains bounded by $2B(\bar{s})$. By extracting a subsequence such that $S_\alpha(s) \to S^*$ we obtain easily $H_s(S^*) = 0$ and also

$$\gamma_\alpha(s) \to \gamma(s) = \inf_{\eta > s} \int_s^\eta H_s(\xi)d\xi = \int_s^{S^*} H_s(\xi)d\xi$$

We call again $S(s)$ the first point such that $H_s(S(s)) = 0$.

We have shown above, in the proof of Theorem 15.1 that

$$\gamma'_\alpha(s) \geq -\frac{Q_\alpha(s_1)}{\beta_{0\alpha}}(1 - \exp -\beta_{0\alpha}(x_{0\alpha} - s_1)) > 0,$$

whenever $s < s_1 < x_{0\alpha}$. Let us fix $s_1 < x_0$. For α sufficiently small we can assume $s_1 < x_{0\alpha}$ and for $s < s_1$

$$\gamma'_\alpha(s) \geq -\frac{Q(s_1)}{2\beta_0}(1 - \exp -\beta_0(x_0 - s_1)) = a(s_1) > 0,$$

and thus

$$\gamma_\alpha(s_1) - \gamma_\alpha(s) \geq (s_1 - s)a(s_1).$$

Since $\gamma_\alpha(s_1) < 0$, in fact

$$-\gamma_\alpha(s) \geq (s_1 - s)a(s_1).$$

Consider now s_α. Either $s_\alpha \geq s_1$ or $s_\alpha < s_1$ in which case from the inequality above

$$K \geq (s_1 - s_\alpha)a(s_1).$$

So in all cases

$$s_\alpha \geq s_1 - \frac{K}{a(s_1)} = \bar{s}. \tag{15.5.18}$$

But then from (15.5.16) we can also assert that

$$S_\alpha = S_\alpha(s_\alpha) \leq 2B(\bar{s}), \tag{15.5.19}$$

for α sufficiently small. Therefore we can extract a subsequence such that $s_\alpha \to s,\ S_\alpha \to S$. Also $\gamma(s) = -K$. However, for the same reasons as for s_α, there is a unique s which satisfies this relation. Hence the full sequence $s_\alpha \to s$. Define

$$\rho_\alpha = \alpha G_{s_\alpha}(s_\alpha) \to \rho = \frac{\sigma^2}{2} Q(s) + f(s) + c\nu + c\lambda\bar{\xi}. \tag{15.5.20}$$

We can then set

$$\tilde{G}_\alpha(x) = G_\alpha(x) - \frac{\rho_\alpha}{\alpha},$$

and claim the following ergodic result

Theorem 15.3. *We assume* (15.2.4). *The function* $\tilde{G}_\alpha(x)$, *(respectively its derivative) converges uniformly on compact sets to* $G(x)$, *(respectively its derivative) and the pair* $G(x), \rho$ *is the solution of the Q.V.I.*

$$\begin{gathered} AG(x) + \rho \leq l(x) \\ G(x) \leq K + \inf_{\eta > x} G(\eta) \\ (AG(x) + \rho - l(x))(G(x) - K + \inf_{\eta > x} G(\eta)) = 0, \end{gathered} \tag{15.5.21}$$

with

$$l(x) = f(x) + c\nu + c\lambda\bar{\xi}.$$

The function $G(x)$ *is* C^1, *its derivative has linear growth. It is given by an* s, S *policy*

$$\begin{gathered} AG(x) + \rho = l(x),\ x > s \\ G(x) = K + \inf_{\eta > s} G(\eta),\ x \leq s \end{gathered} \tag{15.5.22}$$

15.6. PROBABILISTIC INTERPRETATION

We will develop in this section the probabilistic counterpart of the Q.V.I approach of the previous sections. We extend the approach of **[32]** to the diffusion and Poisson case. The idea is to optimize in the class of s, S policies and to show that we solve also the Q.V.I in this way.

15.6.1. STUDY OF THE PROCESS WITH AN s, S CONTROL.

Let $s < S$ two given numbers. We will define the process $y_x^{s,S}(t)$ subject to an s, S policy, with initial value x. Recalling the demand

$$D(t) = \nu t + N(t) + \sigma w(t),$$

in the interval $0, t$ we define

$$\tau_s^x = \inf\{t \geq 0 | x - D(t) \leq s\},$$

and if $x > s$

$$y_x^{s,S}(t) = x - D(t), \forall t < \tau_s^x.$$

Next we set

$$y_x^{s,S}(\tau_s^x) = S.$$

We set

$$\tau_{x,1}^{s,S} = \tau_s^x,$$

and suppose we have defined $\tau_{x,n}^{s,S}$ and $y_x^{s,S}(\tau_{x,n}^{s,S}) = S$. We then define

$$\tau_{x,n+1}^{s,S} = \tau_{x,n}^{s,S} + \inf\{t \geq 0 | S - D(\tau_{x,n}^{s,S} + t) + D(\tau_{x,n}^{s,S}) \leq s\}.$$

Clearly $\tau^{s,S}_{x,n+1} > \tau^{s,S}_{x,n}$. We next set

$$y^{s,S}_x(\tau^{s,S}_{x,n} + t) = S - D(\tau^{s,S}_{x,n} + t) + D(\tau^{s,S}_{x,n}),\ \forall t < \tau^{s,S}_{x,n+1} - \tau^{s,S}_{x,n},$$

and again $y^{s,S}_x(\tau^{s,S}_{x,n+1}) = S$. We have in this way defined a process which is controlled by an s, S policy. It has jumps at times $\tau^{s,S}_{x,n}$, the impulse times and the size of the of the impulses at time $\tau^{s,S}_{x,n}$ is $S - y^{s,S}_x(\tau^{s,S}_{x,n} - 0)$. Note that this impulse is larger or equal to $S - s$. The process $y^{s,S}_x(t)$ has also jumps due to the compound Poisson part of the demand. It is a Markov process for which we are going to describe the infinitesimal generator.

Let us recall that the infinitesimal generator is the linear operator defined on the set of functions $\varphi(x)$ such that the limit

$$\lim_{\epsilon\to 0} \frac{E\varphi(y^{s,S}_x(\epsilon)) - \varphi(x)}{\epsilon} = \mathcal{A}^{s,S}\varphi(x),$$

is defined. Since

$$y^{s,S}_x(\epsilon) = y^{s,S}_S(\epsilon),\ \forall x \le s,$$

we need to have

$$\varphi(x) = \varphi(S),\ \forall x \le s. \tag{15.6.1}$$

Performing the same approximate reasoning done in section 15.2, we obtain easily

$$\mathcal{A}^{s,S}\varphi(x) = \begin{cases} -A\varphi(x), & \text{if } x > s \\ -A\varphi(S), & \text{if } x \le s \end{cases},$$

in which A is the second order differential operator defined in equation (17.2.4). Its domain is the set of functions which are bounded with bounded first and second derivatives and which satisfy (15.6.1). It is sufficient to assume that the functions φ and $A\varphi$ are bounded and that (15.6.1) is satisfied. Although we refer to the domain of $\mathcal{A}$ it will be convenient to speak of the domain of A.

15.6.2. INVARIANT MEASURE. We are going to show the following

Theorem 15.4. *The process $y^{s,S}_x(t)$ has a unique invariant measure defined as follows*

$$m^{s,S}(x) = \frac{\tilde{m}^{s,S}(x)}{\int_s^{+\infty} \tilde{m}^{s,S}(\xi)d\xi},$$

with

$$\tilde{m}^{s,S}(x) = \begin{cases} 0, & \textit{if } x \le s \\ \exp\beta_2 x\ \int_s^x \exp -\beta_2\xi\Gamma(S-\xi)d\xi, & \textit{if } s \le x \le S \\ \exp\beta_2 x\ \int_s^S \exp -\beta_2\xi\Gamma(S-\xi)d\xi, & \textit{if } S \le x \end{cases}, \tag{15.6.2}$$

in which $\Gamma(x)$ is the unique solution of

$$\begin{aligned} -\frac{\sigma^2}{2}\Gamma"(x) + \nu\Gamma'(x) + \lambda\Gamma(x) - \lambda\int_0^x \Gamma(x-\xi)\mu(\xi)d\xi &= 0 \\ \Gamma(0) = 1 \quad 0 \le \Gamma(x) \le 1 & \end{aligned} \tag{15.6.3}$$

and β_2 is the unique negative root of

$$-\frac{\sigma^2}{2}\beta_2^2 - \nu\beta_2 + \lambda - \lambda\hat{\mu}(\beta_2) = 0.$$

Moreover for any $\varphi(x)$ *bounded*

$$\Phi_\alpha^{s,S}(x) = E\int_0^{+\infty} \exp -\alpha t\ \varphi(y_x^{s,S}(t))dt, \tag{15.6.4}$$

satisfies

$$\alpha\Phi_\alpha^{s,S}(x) \to \int_s^{+\infty} \varphi(x)m^{s,S}(x)dx. \tag{15.6.5}$$

PROOF. We note that $\tilde{m}^{s,S}(x)$ in formula (15.6.2) is continuous but not C^1. The derivative is discontinuous in S. We begin to prove that $m^{s,S}(x)$ is an invariant measure and the unique one, which is continuous and C^1 except for S. We begin by proving that $m^{s,S}(x)$ must vanish on the interval $(-\infty, s)$. Indeed an invariant measure must satisfy

$$\int m^{s,S}(x)\varphi(x)dx = \int m^{s,S}(x)E\varphi(y_x^{s,S}(\epsilon))dx, \forall\epsilon$$

Take a test function $\varphi(x)$ which vanishes on $s, +\infty$. We get from the preceding relation

$$\int_{-\infty}^s m^{s,S}(x)\varphi(x)dx = E\varphi(y_S^{s,S}(\epsilon))\int_{-\infty}^s m^{s,S}(x)dx + \int_s^{+\infty} m^{s,S}(x)E\varphi(y_x^{s,S}(\epsilon))dx$$

Letting $\epsilon \to 0$ in the right hand side, we obtain

$$\varphi(S)\int_{-\infty}^s m^{s,S}(x)dx + \int_s^{+\infty} m^{s,S}(x)\varphi(x)dx = 0,$$

from the choice of φ. Let now pick $\varphi(x)$ in the domain of the infinitesimal generator. An invariant measure must satisfy

$$\int_s^{+\infty} m^{s,S}(x)A\varphi(x)dx = 0.$$

We write $m(x)$ for $m^{s,S}(x)$ to simplify notation. We recall the regularity properties on m and perform integration by parts. We get

$$-\frac{\sigma^2}{2}[\varphi(S)(m'(S+0) - m'(S-0)) + \varphi(s)m'(s)]$$
$$-\lambda\varphi(S)\int_s^{+\infty} m(x)\left(\int_{x-s}^{+\infty}\mu(\xi)d\xi\right)dx + \int_s^{+\infty}\varphi(x)\left(-\frac{\sigma^2}{2}m''(x) - \nu m'(x) + \lambda m(x)\right)dx$$
$$-\lambda\int_s^{+\infty} m(x)\left(\int_0^{x-s}\varphi(x-\xi)\mu(\xi)d\xi\right)dx = 0.$$

Using the fact that φ satisfies $\varphi(s) = \varphi(S)$ and rearranging we obtain

$$-\varphi(S)\left[\frac{\sigma^2}{2}(m'(S+0) - m'(S-0) + m'(s))\right.$$
$$\left.+\lambda\int_s^{+\infty} m(x)\left(\int_{x-s}^{+\infty}\mu(\xi)d\xi\right)dx\right] + \int_s^{+\infty}\varphi(x)\left(-\frac{\sigma^2}{2}m''(x) - \nu m'(x)\right.$$
$$\left.+\lambda m(x) - \lambda\int_0^{+\infty} m(x+\xi)\mu(\xi)d\xi\right)dx = 0.$$

This implies

$$-\frac{\sigma^2}{2}m''(x) - \nu m'(x) + \lambda m(x) - \lambda \int_0^{+\infty} m(x+\xi)\mu(\xi)d\xi = 0, x \in (s, +\infty)$$

$$m(x) = 0, x \leq s \quad m' \text{discontinuous in } S \tag{15.6.6}$$

and we must have $m \geq 0$ with the normalisation property $\int_s^{+\infty} m(x)dx = 1$. Note that the condition

$$\frac{\sigma^2}{2}(m'(S+0) - m'(S-0) + m'(s)) + \lambda \int_s^{+\infty} m(x) \left(\int_{x-s}^{+\infty} \mu(\xi)d\xi \right) dx = 0, \tag{15.6.7}$$

is not an additional condition. It follows from (15.6.6) and the normalisation properties. So (15.6.6) and the normalisation properties constitute the equation for the invariant measure. Let us prove that (15.6.2) is a solution and the unique one. We can check that $\exp \beta_2 x$ satisfies the integro-differential equation (15.6.6). It goes to 0 as $x \to +\infty$. It cannot be the solution on $(s, +\infty)$, but if we impose the value at S equal to $\exp \beta_2 S$ then it is the unique solution on $(S, +\infty)$. Note that imposing the normalisation condition or the value at a point are equivalent. So we may fix the value at S equal to $\exp \beta_2 S$. There can then be only one solution of (15.6.6) which is C^2 on $(S, +\infty)$ and vanishes at ∞. This follows from maximum principle considerations.

We can then write the equation on (s, S) as follows

$$\begin{aligned}
&-\frac{\sigma^2}{2}m''(x) - \nu m'(x) + \lambda m(x) - \lambda \int_0^{S-x} m(x+\xi)\mu(\xi)d\xi \\
&\qquad = \lambda \exp \beta_2 x \int_{S-x}^{+\infty} \exp \beta_2 \xi \, \mu(\xi)d\xi, \quad s < x < S \\
&m(s) = 0; \quad m(S) = \exp \beta_2 S
\end{aligned}$$

This equation can have only one solution, from maximum principle considerations. If we make the change of unknown function

$$m(x) = v(x) \exp \beta_2 x,$$

then we obtain an equation for $v(x)$, namely

$$\begin{aligned}
&-\frac{\sigma^2}{2}v''(x) - (\nu + \beta_2\sigma^2)v'(x) + \lambda v(x)\hat{\mu}(\beta_2) \\
&\qquad -\lambda \int_0^{S-x} v(x+\xi) \exp \beta_2 \xi \, \mu(\xi)d\xi = \lambda \int_{S-x}^{+\infty} \exp \beta_2 \xi \, \mu(\xi)d\xi, \\
&v(s) = 0; \quad v(S) = 1
\end{aligned}$$

Considering $w(x) = v'(x)$ we get, taking into account the value $v(S) = 1$

$$-\frac{\sigma^2}{2}w''(x) - (\nu + \beta_2\sigma^2)w'(x) + \lambda w(x)\hat{\mu}(\beta_2) - \lambda \int_0^{S-x} w(x+\xi) \exp \beta_2 \xi \, \mu(\xi)d\xi = 0,$$

and if we write

$$w(x) = \exp -\beta_2 x \; z(S-x),$$

we see that $z(x) = \Gamma(x)$. Collecting results, we obtain that $\tilde{m}^{s,S}(x)$ is a solution of (15.6.6) with all conditions satisfied, with of course a different value at S. Therefore $m^{s,S}(x)$ is the unique invariant measure satisfying the regularity properties, the boundary conditions and the normalization property.

Let us now prove (15.6.5). We have

$$\Phi_\alpha^{s,S}(x) = E\int_0^{\tau_s^x} \exp -\alpha t\ \varphi(y_x^{s,S}(t))dt$$
$$+\sum_{n=1}^{n+1} E\int_{\tau_{x,n}^{s,S}}^{\tau_{x,n+1}^{s,S}} \exp -\alpha t\ \varphi(y_x^{s,S}(t))dt.$$

Then using conditioning and independence properties, for $n \geq 1$

$$E\int_{\tau_{x,n}^{s,S}}^{\tau_{x,n+1}^{s,S}} \exp -\alpha t\ \varphi(y_x^{s,S}(t))dt$$
$$= E\exp -\alpha\tau_{x,n}^{s,S} \int_0^{\tau_{x,n+1}^{s,S}-\tau_{x,n}^{s,S}} \exp -\alpha t\ \varphi(y_x^{s,S}(\tau_{x,n}^{s,S}+t))dt$$
$$= E\exp -\alpha\tau_{x,n}^{s,S}\ E\int_0^{\tau_s^S} \exp -\alpha t\ \varphi(y_S^{s,S}(t))dt$$
$$= E\exp -\alpha\tau_s^x(E\exp -\alpha\tau_s^S)^{n-1}\ E\int_0^{\tau_s^S} \exp -\alpha t\ \varphi(y_S^{s,S}(t))dt.$$

We obtain the formula

$$\Phi_\alpha^{s,S}(x) = E\int_0^{\tau_s^x} \exp -\alpha t\ \varphi(y_x^{s,S}(t))dt$$

(15.6.8)
$$+E\exp -\alpha\tau_s^x\ \frac{E\int_0^{\tau_s^S} \exp -\alpha t\ \varphi(y_S^{s,S}(t))dt}{1-E\exp -\alpha\tau_s^S}.$$

Then

(15.6.9)
$$\alpha\Phi_\alpha^{s,S}(x) \to \frac{E\int_0^{\tau_s^S} \varphi(y_S^{s,S}(t))dt}{E\tau_s^S}.$$

If we define

$$\zeta_s(x;\varphi) = E\int_0^{\tau_s^x} \varphi(y_x^{s,S}(t))dt,$$

then we have proven that

$$\alpha\Phi_\alpha^{s,S}(x) \to \frac{\zeta_s(S;\varphi)}{\zeta_s(S;1)}.$$

However $\zeta_s(x;\varphi)$ is the solution of the problem

(15.6.10)
$$\begin{aligned} A\zeta_s(x) &= \varphi(x) \quad x > s \\ \zeta_s(x) &= 0 \quad x \leq s \end{aligned}$$

and it has the explicit solution

(15.6.11)
$$\zeta_s(x;\varphi) = \frac{2}{\sigma^2}\int_s^x \Gamma(x-\xi)\int_\xi^{+\infty} \exp\beta_2(\eta-\xi)\varphi(\eta)d\eta.$$

We deduce, after some easy changes of integration

$$\zeta_s(S;\varphi) = \frac{2}{\sigma^2}\int_s^S \varphi(\eta)\exp\beta_2\eta\left(\int_s^\eta \exp -\beta_2\xi\Gamma(S-\xi)d\xi\right)d\eta$$
$$+\frac{2}{\sigma^2}\int_S^{+\infty} \varphi(\eta)\exp\beta_2\eta\left(\int_s^S \exp -\beta_2\xi\Gamma(S-\xi)d\xi\right)d\eta$$

and we recognize as expected

$$\zeta_s(S;\varphi) = \frac{2}{\sigma^2}\int_s^{+\infty} \varphi(x)\tilde{m}^{s,S}(x)dx.$$

The result follows. □

We now present a useful result

Lemma 15.2. *For any $z \in$ domain of A, we have the equality*

$$\int_s^{+\infty} m(x)z(x)Az(x)dx = \frac{1}{2}\int_s^{+\infty} \sigma^2 m z'^2 dx$$
$$+\frac{\lambda}{2}\int_s^{+\infty}\int_0^{+\infty} m(x)(z(x)-z(x-\xi))^2\mu(\xi)dxd\xi. \tag{15.6.12}$$

PROOF. We consider the quantity on the left hand side. Since we will perform two integration by parts, we have to be careful that m' is discontinuous in S. On the other hand z' is continuous in S.

After one integration by parts we can write

$$\int_s^{+\infty} m(x)z(x)Az(x)dx = \frac{1}{2}\int_s^{+\infty} \sigma^2 m z'^2 dx + \int_s^S zz'\left(\frac{\sigma^2}{2}m' + \nu m\right)dx$$
$$+\lambda\int_s^{+\infty}\int_0^{+\infty} m(x)z(x)(z(x)-z(x-\xi))\mu(\xi)dxd\xi.$$

Noting that

$$zz' = \frac{1}{2}(z^2)',$$

we can perform a second integration by parts which leads to

$$\int_s^S zz'\left(\frac{\sigma^2}{2}m' + \nu m\right)dx = -\frac{1}{2}\int_s^{+\infty} z^2\left(\frac{\sigma^2}{2}m" + \nu m'\right)dx +$$
$$+\frac{\sigma^2}{4}z^2(S)(m'(S-0) - m'(S+0) - m'(s)).$$

The next step is to consider

$$\int_s^{+\infty}\int_0^{+\infty} m(x)z(x)(z(x)-z(x-\xi))\mu(\xi)dxd\xi$$
$$= \frac{1}{2}\int_s^{+\infty}\int_0^{+\infty} m(x)(z(x)-z(x-\xi))^2\mu(\xi)dxd\xi$$
$$+\frac{1}{2}\int_s^{+\infty}\int_0^{+\infty} m(x)(z^2(x)-z^2(x-\xi))\mu(\xi)dxd\xi.$$

We also use

$$\int_s^{+\infty}\int_0^{+\infty} m(x)z^2(x-\xi)\mu(\xi)dxd\xi = \int_s^{+\infty}\int_0^{+\infty} m(x+\xi)z^2(x)\mu(\xi)dxd\xi$$
$$+z^2(S)\int_s^{+\infty}\int_0^{+\infty} m(x+\xi)\mu(\xi)dxd\xi.$$

Collecting results and making use of the equation of the invariant measure, see (15.6.6), (15.6.7) we obtain many simplifications. Eventually we obtain (15.6.14) □

The function $\Phi_\alpha^{s,S}(x)$ (we drop the notation s, S) defined in (15.6.4) is the solution of

$$\begin{aligned} A\Phi_\alpha(x) + \alpha\Phi_\alpha(x) &= \varphi(x), \quad \forall x > s \\ \Phi_\alpha(x) &= \Phi_\alpha(S), \quad \forall x \leq s \end{aligned}$$

The solution $\Phi_\alpha(x)$ belongs to the domain of A. It will be useful to prove a density result.

Lemma 15.3. *For any bounded function φ we can construct a sequence φ_ϵ belonging to the domain of A such that*

$$||\varphi_\epsilon|| = \sup_x |\varphi_\epsilon(x)| \leq ||\varphi||$$

$$\int_s^{+\infty} m(x)|\varphi_\epsilon(x) - \varphi(x)|^2 dx \to 0, \quad as \quad \epsilon \to 0 \tag{15.6.13}$$

Proof. We consider the solution of the problem

$$\begin{aligned} A\varphi_\epsilon(x) + \frac{\varphi_\epsilon(x)}{\epsilon} &= \frac{\varphi(x)}{\epsilon}, \quad \forall x > s \\ \varphi_\epsilon(x) &= \varphi_\epsilon(S), \quad \forall x \leq s \end{aligned}$$

Set

$$\tilde{\varphi}_\epsilon(x) = \varphi_\epsilon(x) - ||\varphi||,$$

then

$$\begin{aligned} A\tilde{\varphi}_\epsilon(x) + \frac{\tilde{\varphi}_\epsilon(x)}{\epsilon} &= \frac{\varphi(x) - ||\varphi||}{\epsilon}, \quad \forall x > s \\ \tilde{\varphi}_\epsilon(x) &= \tilde{\varphi}_\epsilon(S), \quad \forall x \leq s \end{aligned}$$

and thus $\tilde{\varphi}_\epsilon(x) \leq 0$. Similarly

$$\varphi_\epsilon(x) + ||\varphi|| \geq 0,$$

and the first property (15.6.22) follows. From Lemma 15.2 we can assert that

$$\frac{1}{2}\int_s^{+\infty} \sigma^2 m(\varphi_\epsilon)^2 dx + \frac{\lambda}{2}\int_s^{+\infty}\int_0^{+\infty} m(x)(\varphi_\epsilon(x) - \varphi_\epsilon(x-\xi))^2 \mu(\xi) dx d\xi$$
$$+ \int_s^{+\infty} m\frac{\varphi_\epsilon^2}{\epsilon} dx = \int_s^{+\infty} m\frac{\varphi\varphi_\epsilon}{\epsilon} dx.$$

It follows that

$$\int_s^{+\infty} m\varphi_\epsilon^2 dx \leq \int_s^{+\infty} m\varphi_\epsilon f dx.$$

Consider the Hilbert space $L_m^2(R)$ of functions z such that

$$\int_s^{+\infty} mz^2 dx < +\infty.$$

The sequence φ_ϵ remains bounded in this Hilbert space. If we consider a weakly convergent subsequence converging towards φ^* then

$$\int_s^{+\infty} m(\varphi^*)^2 dx \leq \limsup \int_s^{+\infty} m\varphi_\epsilon^2 dx \leq \lim \int_s^{+\infty} m\varphi_\epsilon\varphi dx = \int_s^{+\infty} m\varphi^*\varphi dx.$$

Now

$$\int_s^{+\infty} m(\varphi_\epsilon - \varphi^*)^2 dx \leq \int_s^{+\infty} m\varphi_\epsilon\varphi dx + \int_s^{+\infty} m(\varphi^*)^2 dx - 2\int_s^{+\infty} m\varphi_\epsilon\varphi^* dx,$$

hence

$$\limsup \int_s^{+\infty} m(\varphi_\epsilon - \varphi^*)^2 dx \le \int_s^{+\infty} m\varphi^*\varphi dx - \int_s^{+\infty} m(\varphi^*)^2 dx. \tag{15.6.14}$$

Noting that $\sqrt{\epsilon}\varphi_\epsilon'$ remains bounded in $L^2_m(R)$, it follows from the equation of φ_ϵ that

$$\epsilon \int_s^{+\infty} A\varphi_\epsilon z dx \to 0, \quad \forall z \quad \text{smooth with compact support in } (s, +\infty),$$

therefore we deduce immediately that $\varphi^* = \varphi$ and from (15.6.14) the property (15.6.22) is obtained, which completes the proof. □

15.6.3. ERGODICITY. The result of Theorem 15.4, although sufficient to study ergodic control and the limit of Bellman equation, as we shall see in the next sections, does not prove full ergodicity. To complete we need to prove the

Theorem 15.5. *We have the property*

$$E\varphi(y_x^{s,S}(t)) \to \int_s^{+\infty} \varphi(\xi) m^{s,S}(\xi) d\xi, \text{ as } t \to +\infty, \forall x, \ \forall \varphi \text{ bounded.} \tag{15.6.15}$$

PROOF. We shall use the fact that

$$z^{s,S}(x,t) = E\varphi(y_x^{s,S}(t)),$$

is the solution of the Cauchy problem

$$\begin{aligned} &\frac{\partial z}{\partial t} + Az = 0, \quad x > s \\ &z(x,t) = z(S,t), \quad x \le s, t > 0 \\ &z(x,0) = \varphi(x) \end{aligned} \tag{15.6.16}$$

We proceed with estimates. We first derive an energy equality. By testing (15.6.16) with mz and making use of Lemma 15.2 we can state

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_s^{+\infty} m(x)z^2(x,t)dx + \frac{1}{2}\int_s^{+\infty} \sigma^2 m \left(\frac{\partial z}{\partial x}\right)^2 dx \\ &\quad + \frac{\lambda}{2}\int_s^{+\infty}\int_0^{+\infty} m(x)(z(x) - z(x-\xi))^2 \mu(\xi) dx d\xi = 0 \end{aligned} \tag{15.6.17}$$

We deduce the estimates

$$\int_0^{+\infty}\int_s^{+\infty} \sigma^2 m \left(\frac{\partial z}{\partial x}\right)^2 (x,t) dx dt \le \int_s^{+\infty} m(x)\varphi^2(x) dx \tag{15.6.18}$$

(15.6.19)

$$\lambda \int_0^{+\infty}\int_s^{+\infty}\int_0^{+\infty} m(x)(z(x) - z(x-\xi))^2 \mu(\xi) d\xi dx dt \le \int_s^{+\infty} m(x)\varphi^2(x) dx.$$

Also

$$\int_s^{+\infty} m(x) z^2(x,t) dx \le \int_s^{+\infty} m(x)\varphi^2(x) dx, \quad \forall t. \tag{15.6.20}$$

This inequality implies that without loss of generality we may assume that φ is in the domain of A. Indeed considering the approximation φ_ϵ defined in Lemma

15.3 and noting by $z_\epsilon(x,t)$ the corresponding solution of the Cauchy problem, we deduce from the estimate (15.6.20) that

$$\int_s^{+\infty} m(x)(z_\epsilon(x,t)-z(x,t))^2dx \leq \int_s^{+\infty} m(x)(\varphi_\epsilon-\varphi)^2(x)dx.$$

Suppose we have proven that

$$z_\epsilon(x,t) \to \int_s^{+\infty} m(x)\varphi_\epsilon(x)dx, \quad \text{as} \quad t\to+\infty,\ \forall\epsilon>0$$

then writing

$$\bar\varphi = \int_s^{+\infty} m(x)\varphi(x)dx, \quad \bar\varphi_\epsilon = \int_s^{+\infty} m(x)\varphi_\epsilon(x)dx$$

$$\begin{aligned}
&\int_s^{+\infty} m(x)(z(x,t)-\bar\varphi)^2dx\\
&\leq 3\left[\int_s^{+\infty} m(x)(z_\epsilon(x,t)-z(x,t))^2dx+\int_s^{+\infty} m(x)(z_\epsilon(x,t)-\bar\varphi_\epsilon)^2dx+(\bar\varphi_\epsilon-\bar\varphi)^2\right]\\
&\leq 3\left[\int_s^{+\infty} m(x)(\varphi_\epsilon-\varphi)^2(x)dx+\int_s^{+\infty} m(x)(z_\epsilon(x,t)-\bar\varphi_\epsilon)^2dx+(\bar\varphi_\epsilon-\bar\varphi)^2\right]
\end{aligned}$$

but then

$$\limsup_{t\to+\infty}\int_s^{+\infty} m(x)(z(x,t)-\bar\varphi)^2dx \leq 3\left[\int_s^{+\infty} m(x)(\varphi_\epsilon-\varphi)^2(x)dx+(\bar\varphi_\epsilon-\bar\varphi)^2\right],$$

and letting $\epsilon\to 0$ we obtain

$$\int_s^{+\infty} m(x)(z(x,t)-\bar\varphi)^2dx \to 0, \text{ as } \quad t\to+\infty$$

We deduce also

$$z(x,t)\to\bar\varphi \quad \text{a.e. } x>s$$

So we assume that φ belongs to the domain of A. We can then obtain further estimates.

We define the function

$$u(x,t)=\frac{\partial z}{\partial t}(x,t),$$

which is solution of

$$\begin{aligned}
&\frac{\partial u}{\partial t}+Au=0, && x>s\\
&u(x,t)=u(S,t), && x\leq s, t>0 \qquad (15.6.21)\\
&u(x,0)=-A\varphi
\end{aligned}$$

As for z we can write

$$\int_0^{+\infty}\int_s^{+\infty}\sigma^2 m\left(\frac{\partial u}{\partial x}\right)^2(x,t)dxdt \leq \int_s^{+\infty} m(x)(A\varphi)^2(x)dx \qquad (15.6.22)$$

$$\begin{aligned}
&\lambda\int_0^{+\infty}\int_s^{+\infty}\int_0^{+\infty} m(x)(u(x)-u(x-\xi))^2\mu(\xi)d\xi dxdt \qquad (15.6.23)\\
&\leq \int_s^{+\infty} m(x)(A\varphi)^2(x)dx
\end{aligned}$$

$$\int_s^{+\infty} m(x)u^2(x,t)dx \leq \int_s^{+\infty} m(x)(A\varphi)^2(x)dx, \quad \forall t. \tag{15.6.24}$$

In particular we have shown that

$$\int_0^{+\infty}\int_s^{+\infty} m\left(\frac{\partial^2 z}{\partial x\partial t}\right)^2(x,t)dxdt \leq \frac{1}{\sigma^2}\int_s^{+\infty} m(x)(A\varphi)^2(x)dx$$

Noting that

$$\int_s^{+\infty} m\left(\frac{\partial z}{\partial x}\right)^2(x,t)dx = \int_s^{+\infty} m(\varphi')^2(x)dx + 2\int_0^t\int_s^{+\infty} m\frac{\partial z}{\partial x}(x,\tau)\frac{\partial^2 z}{\partial x\partial t}(x,\tau)dxd\tau$$

the right hand side has a limit as $t \to +\infty$, thanks to the estimates above. So $\int_s^{+\infty} m\left(\frac{\partial z}{\partial x}\right)^2(x,t)dx$ has a limit as $t \to +\infty$. This limit must be 0 thanks to the estimate (15.6.18). Recalling that z is bounded, we can state that for any compact subset of $[s,+\infty)$ we can find a subsequence of $z(x,t)$ which converges uniformly. The limit is necessarily a constant and since

$$\int_s^{+\infty} m(x)z(x,t)dx = \int_s^{+\infty} m(x)\varphi(x)dx,$$

this limit is necessarily $\int_s^{+\infty} m(x)\varphi(x)dx$. By the uniqueness of the limit, the full sequence converges uniformly on any compact. So we have obtained the result for φ in the domain of A and also the general result, by the approximation above. However we have proven the result (15.6.15) only a.e. x. We want to prove now that it is true for any initial value x. We will need two important intermediary results. The first one is

$$z(x,t) \text{ is continuous in } x, \forall t > 0 \text{ 15.6.43} \tag{15.6.25}$$

This property allows us to assume with no loss of generality that $\varphi(x)$ is continuous and bounded. Indeed, clearly

$$\lim_{t\to+\infty} z(x,t) = \lim_{t\to+\infty} z(x,t+1),$$

and $z(x,t+1)$ satisfies the same equation as $z(x,t)$ except for the initial condition which is $z(x,1)$ instead of $\varphi(x)$. But then $z(x,1)$ is continuous and bounded. The second important result is that in the case when $\varphi(x)$ is continuous and bounded, we can improve the approximation result of Lemma 15.3 as follows: The approximation $\varphi_\epsilon(x)$ defined in Lemma 15.3 satisfies

$$\sup_{s\leq x\leq M} |\varphi_\epsilon(x) - \varphi(x)| \to 0, \text{ as } \epsilon \to 0, \forall M \geq s. \tag{15.6.26}$$

We will also prove the following regularity property for $z(x,t)$

$$\int_s^{+\infty} m^3(x)\left(\frac{\partial z}{\partial x}\right)^2(x,t)dx \leq C\frac{t+1}{t}||\varphi||^2, \forall t > 0. \tag{15.6.27}$$

Note that here φ is only measurable and bounded. When φ belongs to the domain of A we have seen above that a better estimate holds. As a consequence we will have, noting $y_x(t)$ the process $y_x^{s,S}(t)$

$$\sup_{t\geq\delta}\sup_{s_0\leq x\leq M} E\mathbb{1}_{y_x(t)\geq R} \to 0, \text{ as } \quad R \to +\infty, \forall\delta > 0, s < s_0 \leq M. \tag{15.6.28}$$

The intermediary results (15.6.43), (15.6.26), (15.6.27), (15.6.28) will be proved later. We now assume $\varphi(x)$ continuous and bounded. We want to prove that

$$z(x,t) \to \int_s^{+\infty} m(x)\varphi(x)dx \text{ as } t \to +\infty, \forall x. \tag{15.6.29}$$

It is sufficient to assume $x > s$. So let us pick $x_0 > s$. Using $z(x,t) = E\varphi(y_x(t))$ we can write

$$E\varphi(y_{x_0}(t)) - \int m\varphi dx = E\varphi(y_{x_0}(t)) - E\varphi_\epsilon(y_{x_0}(t)) + E\varphi_\epsilon(y_{x_0}(t)) - \int m\varphi_\epsilon dx + \int m\varphi_\epsilon dx - \int m\varphi dx.$$

Now for $R > x_0$ we have

$$|E\varphi(y_{x_0}(t)) - E\varphi_\epsilon(y_{x_0}(t))| \leq 2||\varphi|| E\mathbb{1}_{y_x(t)\geq R} + \sup_{s\leq x\leq R} |\varphi_\epsilon(x) - \varphi(x)|,$$

hence

$$\left|E\varphi(y_{x_0}(t)) - \int m\varphi dx\right| \leq 2||\varphi|| E\mathbb{1}_{y_x(t)\geq R} + \sup_{s\leq x\leq R} |\varphi_\epsilon(x) - \varphi(x)| + \left|E\varphi_\epsilon(y_{x_0}(t)) - \int m\varphi_\epsilon dx\right| + \left|\int m\varphi_\epsilon dx - \int m\varphi dx\right|.$$

Since φ_ϵ belongs to the domain of A, fixing ϵ and R, we can let $t \to +\infty$ and deduce

$$\limsup_{t\to+\infty} \left|E\varphi(y_{x_0}(t)) - \int m\varphi dx\right| \leq 2||\varphi|| \sup_{t\geq\delta} E\mathbb{1}_{y_x(t)\geq R} + \sup_{s\leq x\leq R} |\varphi_\epsilon(x) - \varphi(x)| + \left|\int m\varphi_\epsilon dx - \int m\varphi dx\right|.$$

But then letting $\epsilon \to 0$, then $R \to +\infty$ and making use of properties (15.6.26), (15.6.28) the right hand side tends to 0. So the property (15.6.15) is fully proven.

PROOF OF INTERMEDIARY RESULTS:

PROOF OF REGULARITY:

We know that

$$|z(x,t)| \leq ||\varphi||, \quad \int_0^{+\infty}\int_s^{+\infty} m(x)\left(\frac{\partial z}{\partial x}\right)^2(x,t)dxdt \leq \frac{1}{\sigma^2}||\varphi||^2$$
$$\int_0^{+\infty}\int_s^{+\infty}\int_0^{+\infty} m(x)(z(x) - z(x-\xi))^2\mu(\xi)d\xi dxdt \leq \frac{1}{\lambda}||\varphi||^2 \tag{15.6.30}$$

We now test the equation (15.6.16) with $m^3\dfrac{\partial z}{\partial t}\,\dfrac{t}{t+1}$ and obtain

$$\begin{aligned}&\frac{t}{t+1}\int_s^{+\infty} m^3(x)\left(\frac{\partial z}{\partial t}\right)^2(x,t)dx\\&\quad+\frac{t}{t+1}\int_s^{+\infty} m^{\frac{3}{2}}\frac{\partial z}{\partial t}\left[\frac{3}{2}\sigma^2 m^{\frac{1}{2}}m'\frac{\partial z}{\partial x}+\nu m^{\frac{3}{2}}\frac{\partial z}{\partial x}\right](x,t)dx\\&\quad-\lambda\frac{t}{t+1}\int_s^{+\infty} m^3\frac{\partial z}{\partial t}\left[\int_0^{+\infty}(z(x-\xi,t)-z(x,t))\mu(\xi)d\xi\right]dx\\&\quad+\frac{\sigma^2}{4}\frac{t}{t+1}\frac{d}{dt}\int_s^{+\infty} m^3(x)\left(\frac{\partial z}{\partial x}\right)^2(x,t)dx=0\end{aligned}$$

hence

$$\begin{aligned}&\frac{1}{2}\frac{t}{t+1}\int_s^{+\infty} m^3(x)\left(\frac{\partial z}{\partial t}\right)^2(x,t)dx+\frac{\sigma^2}{4}\frac{t}{t+1}\frac{d}{dt}\int_s^{+\infty} m^3(x)\left(\frac{\partial z}{\partial x}\right)^2(x,t)dx\\&\leq\int_s^{+\infty} m\left(\frac{\partial z}{\partial x}\right)^2\left(\frac{3}{2}\sigma^2 m'+\nu m\right)^2(x,t)dx\\&\quad+\lambda^2\int_s^{+\infty} m^3\int_0^{+\infty}(z(x-\xi,t)-z(x,t))^2\mu(\xi)d\xi.\end{aligned}$$

From the estimates already obtained (15.6.30) we deduce (15.6.27) and also

$$\int_0^{+\infty}\frac{t}{t+1}\int_s^{+\infty} m^3(x)\left(\frac{\partial z}{\partial t}\right)^2(x,t)dxdt\leq C||\varphi||^2. \tag{15.6.31}$$

PROOF OF (15.6.27):

Let $x_0\in[s_0,M]$. Take ρ such that

$$0<\rho\leq\frac{s_0-s}{2}.$$

We call

$$z_R(x,t)=E\mathbb{1}_{y_x(t)\geq R}.$$

We use the estimate

$$\int_s^{+\infty} m^3(x)\left(\frac{\partial z_R}{\partial x}\right)^2(x,t)dx\leq C\frac{t+1}{t},$$

and the constant does not depend on R. In particular we deduce

$$\int_{\frac{s+s_0}{2}}^{M+\frac{s_0-s}{2}}\left(\frac{\partial z_R}{\partial x}\right)^2(x,t)dx\leq C_{s_0,M}\left(1+\frac{1}{t}\right), \tag{15.6.32}$$

where we have used the fact that $m(x)$ has a positive infimum on the interval $[\frac{s+s_0}{2},\ M+\frac{s_0-s}{2}]$. Now we have

$$\begin{aligned}\int_s^{+\infty} mz_R(x,t)dx &= \int_s^{+\infty} m\mathbb{1}_R(x)dx=\int_R^{+\infty} m(x)dx\\&\geq\int_{x_0-\rho}^{x_0+\rho} mz_R(x,t)dx\end{aligned}$$

Now for $x \in [x_0 - \rho, x_0 + \rho]$ we have

$$|z_R(x,t) - z_R(x_0,t)| \leq \rho^{\frac{1}{2}} \left(\int_{\frac{s+s_0}{2}}^{M+\frac{s_0-s}{2}} \left(\frac{\partial z_R}{\partial x}\right)^2 (x,t)dx \right)^{\frac{1}{2}} \leq \rho^{\frac{1}{2}} \left(C_{s_0,M}\right)^{\frac{1}{2}} \left(1+\frac{1}{t}\right)^{\frac{1}{2}}.$$

Let now assume $t \geq \delta$ and set

$$C_{s_0,M,\delta} = (C_{s_0,M})^{\frac{1}{2}} \left(1 + \frac{1}{\delta}\right)^{\frac{1}{2}},$$

we can write, using previous estimates

$$\int_R^{+\infty} m(x)dx \geq (z_R(x_0,t) - \rho^{\frac{1}{2}} C_{s_0,M,\delta}) \int_{x_0-\rho}^{x_0+\rho} mdx,$$

and thus

$$z_R(x_0,t) \leq \rho^{\frac{1}{2}} C_{s_0,M,\delta} + \frac{\int_R^{+\infty} m(x)dx}{\int_{x_0-\rho}^{x_0+\rho} mdx},$$

thus also

$$\sup_{x\in[s_0,M]} \sup_{t\geq\delta} z_R(x_0,t) \leq \rho^{\frac{1}{2}} C_{s_0,M,\delta} + \frac{\int_R^{+\infty} m(x)dx}{\inf_{x\in[s_0,M]} \int_{x-\rho}^{x+\rho} m(\xi)d\xi}.$$

Letting $R \to +\infty$, then ρ to 0, we obtain the property (15.6.27). This argument is

PROOF OF CONTINUITY (15.6.43):

We first prove the continuity of $z(x,t)$ for t fixed positive, inside $(s, +\infty)$. It is a direct consequence of the estimate (15.6.27). Indeed for any fixed positive t and any compact $[s_0, M]$ $z(x,t)$ has its partial derivative in x bounded in $L^2(s_0, M)$. Therefore it is a continuous function of x on $[s_0, M]$.

So what remains is to prove the continuity to the right in s, namely

(15.6.33) $$z(s+\epsilon, t) \to z(S,t) \ \forall t > 0, \text{ as } \epsilon \to 0.$$

Since on the left of s, $z(x,t) = z(S,t)$, the property (15.6.33) will imply continuity in x at any point. Recall the notation $\tau_{x,n}$ introduced in section 15.6.1 to characterize the trajectory $y_x(t)$, with $\tau_{x,0} = 0$, $\tau_{x,1} = \tau_s^x$ and

$$\tau_{x,n+1} = \tau_{x,n} + X_{x,n}, \quad n \geq 1,$$

with

$$X_{x,n} = \inf\{u \geq 0 | S - D(\tau_{x,n} + u) + D(\tau_{x,n})\}.$$

The variables $X_{x,n}$ are independent and have an identical probability distribution, that of τ_s^S. If we define

$$F_x(t) = E\mathbb{1}_{\tau_s^x < t},$$

then the probability distribution of $X_{x,n}$ is $F_S(t)$. Note that

$$y_x(t) = \begin{cases} x - D(t), & \text{if } t < \tau_s^x \\ S - D(t) + D(\tau_{x,n}), & \text{if } \tau_{x,n} \leq t < \tau_{x,n+1} \end{cases}$$

We can then obtain a formula for $z(x,t)$, $t > 0$. We have

$$z(x,t) = E\varphi(x - D(t))\mathbb{1}_{t<\tau_s^x} + \sum_{n=1}^{+\infty} E\varphi(S - D(t) + D(\tau_{x,n}))\mathbb{1}_{\tau_{x,n}\leq t<\tau_{x,n+1}}$$

Define

$$\Psi(x,t) = E\varphi(x - D(t))\mathbb{1}_{t<\tau_s^x},$$

then, recalling $\mathcal{F}^t = \sigma(D(s), s \leq t)$, by conditioning with respect to $\mathcal{F}^{\tau_{x,n}}$ and using the independence properties one can check that

$$E\varphi(S - D(t) + D(\tau_{x,n}))\mathbb{1}_{\tau_{x,n}\leq t<\tau_{x,n+1}} = E\mathbb{1}_{\tau_{x,n}\leq t}\Psi(S, t - \tau_{x,n}),\ n \geq 1.$$

Since

$$\tau_{x,n} = \tau_s^x + \sum_{j=1}^{n-1} X_{x,j},\ n \geq 2,$$

using the independence properties, we check easily

$$E\mathbb{1}_{\tau_{x,n}\leq t} = F_x \otimes F_S^{\otimes(n-1)}(t),$$

where $F_S^{\otimes(n-1)}$ denotes the $n-1$ times convolution of the probability distribution F_S. The cumulative distribution function $F_S(t)$ has a density $f_S(t)$ and

$$dF_S^{\otimes(n-1)}(t) = f_S^{\otimes(n-1)}(t)dt.$$

Note that for $n = 1$ there is no density and $F_S^{(0)}(t) = 1, \forall t > 0$. In that case $f_S^{(0)}(t)dt$ must be replaced with the Dirac measure in 0. Collecting results we can finally write the formula

$$z(x,t) = \Psi(x,t) + \sum_{n=1}^{\infty} \int_0^t f_x(t-\eta)\left(\int_0^\eta \Psi(S,\xi) f_S^{\otimes(n-1)}(\eta - \xi)d\xi\right)d\eta \tag{15.6.34}$$

So we have to consider the limit of

$$\begin{aligned} z(s+\epsilon, t) &= \Psi(s+\epsilon, t) + \int_0^t f_{s+\epsilon}(t-\eta)\Psi(S,\eta)d\eta \\ &+ \sum_{n=2}^{\infty} \int_0^t f_{s+\epsilon}(t-\eta)\left(\int_0^\eta \Psi(S,\xi) f_S^{\otimes(n-1)}(\eta-\xi)d\xi\right)d\eta \end{aligned} \tag{15.6.35}$$

However $\Psi(x,t)$ is the solution of the Cauchy problem

$$\begin{aligned} \frac{\partial\Psi}{\partial t} + A\Psi = 0, &\quad x > s \\ \Psi(x,t) = 0, &\quad x \leq s \\ \Psi(x,0) = \varphi(x) & \end{aligned} \tag{15.6.36}$$

This is a classical Cauchy problem with Dirichlet boundary conditions. Note that the solution does not depend on the value of $\varphi(x)$ for $x \leq s$. It is then standard to check that

$$\Psi(s+\epsilon, t) \to 0, \text{ as } \epsilon \to 0,\ \forall t > 0.$$

Now if we take $\varphi(x) = 1$ in (15.6.35) then we see that

$$E\mathbb{1}_{t<\tau_s^{s+\epsilon}} \to 0, \text{ as } \epsilon \to 0,\ \forall t > 0.$$

This implies

$$\mathbb{1}_{t<\tau_s^{s+\epsilon}} \to 0\ a.s. \text{ as } \epsilon \to 0,\ \forall t > 0.$$

It is then easy to check that

$$E\frac{\tau_s^{s+\epsilon}}{1+\tau_s^{s+\epsilon}} \to 0,$$

and also $\tau_s^{s+\epsilon} \to 0,$ a.s. But then we can pass to the limit in formula (15.6.35) and get

$$\begin{aligned} z(s+\epsilon,t) &\to \Psi(S,t)+\sum_{n=2}^{\infty}\int_0^t \Psi(S,\xi) f_S^{\otimes(n-1)}(t-\xi)d\xi \\ &= z(S,t) \end{aligned}$$

the last equality is checked by comparing with (15.6.34).

PROOF OF (15.6.26):

We begin by proving a weaker result. Let $0 < a_0 \le \dfrac{M}{2}$, then we have

(15.6.37) $$\sup_{s+a_0\le x\le M} |\varphi_\epsilon(x)-\varphi(x)| \to 0, \text{ as } \epsilon \to 0.$$

This result needs the continuity of $\varphi(x)$ but not necessarily $\varphi(s)=\varphi(S)$. We use

$$\begin{aligned} \varphi_\epsilon(x) &= \frac{1}{\epsilon}E\int_0^{+\infty} \exp -\frac{t}{\epsilon}\varphi(y_x(t))dt \\ &= E\int_0^{+\infty} \exp -u\ \varphi(y_x(\epsilon u))du \end{aligned}$$

For T fixed set

$$\varphi_\epsilon^T(x) = E\int_0^T \exp -u\ \varphi(y_x(\epsilon u))du,$$

and

(15.6.38) $$|\varphi_\epsilon(x)-\varphi_\epsilon^T(x)| \le ||\varphi|| \exp -T.$$

We next assert that for $u \le T$

$$E\varphi(y_x(\epsilon u)) = E\varphi(x-D(\epsilon u)) + E(\varphi(y_x(\epsilon u)) - \varphi(x-D(\epsilon u)))\mathbb{1}_{\tau_s^x\le\epsilon T},$$

hence for $u \le T$

(15.6.39) $$|E\varphi(y_x(\epsilon u)) - E\varphi(x-D(\epsilon u))| \le 2||\varphi||E\mathbb{1}_{\tau_s^x\le\epsilon T}.$$

It is readily seen that

$$E\mathbb{1}_{\tau_s^x\le\epsilon T} \le P\left(\nu\epsilon T + \sigma \sup_{0\le u\le\epsilon T} w(u) + N(\epsilon T) \ge a_0\right) = \Theta_{a_o}(\epsilon T).$$

This quantity is of course equal to 1 when $\nu\epsilon T \ge a_0$. To evaluate this quantity when $\nu\epsilon T < a_0$ we shall recall the following probability distributions

$$P\left(\sup_{0\le u\le\epsilon T} w(u) \le y\right) = \frac{2}{\sqrt{2\pi\epsilon T}}\int_0^y \exp -\frac{\xi^2}{2\epsilon T}d\xi$$

(see I. Karatzas, S. Shreve [**27**]), and

$$P(N(\epsilon T) \quad \le \quad y) = 1 - \lambda\epsilon T\int_y^{+\infty} \mu(\eta)d\eta + o(\epsilon)$$

But then

$$P\left(N(\epsilon T) + \sigma\sup_{0\le u\le\epsilon T} w(u) \ge y\right) \le \lambda\epsilon T + P\left(\sigma\sup_{0\le u\le\epsilon T} w(u) \ge y\right).$$

Collecting results we obtain

$$\Theta_{a_o}(T\epsilon) \leq \lambda\epsilon T + \frac{2}{\sqrt{2\pi}} \int_{\frac{a_0 - \nu\epsilon T}{\sigma\sqrt{\epsilon T}}}^{+\infty} \exp -\frac{\xi^2}{2} d\xi. \tag{15.6.40}$$

Next if $|D(\epsilon u)| < a_0$ we have $x - D(\epsilon u) \in [s, M]$. Since $\varphi(x)$ is uniformly continuous on $[s, M]$ there exists a function $\gamma_M(\delta)$ which is monotone increasing and tends to 0 as δ goes to 0 and

$$|\varphi(x') - \varphi(x")| \leq \gamma_M(\delta) \text{ if } |x' - x"| \leq \delta.$$

Therefore

$$|\varphi(x - D(\epsilon u)) - \varphi(x)| \leq \gamma_M(|D(\epsilon u)|) + 4||\varphi||\mathbb{1}_{|D(\epsilon u)| \geq a_0}.$$

Since

$$|D(\epsilon u)| \leq \nu\epsilon T + \sigma \sup_{0 \leq u \leq \epsilon T} w(u) + N(\epsilon T),$$

we get

$$E|\varphi(x - D(\epsilon u)) - \varphi(x)| \leq E\gamma_M \left(\nu\epsilon T + \sigma \sup_{0 \leq u \leq \epsilon T} w(u) + N(\epsilon T)\right) + 4||\varphi||\Theta_{a_o}(T\epsilon).$$

Finally we obtain the estimate, for $s + a_0 \leq x \leq M$

$$\begin{aligned} |\varphi_\epsilon(x) - \varphi(x)| \leq 2||\varphi|| \exp -T \ + 6||\varphi||\Theta_{a_o}(T\epsilon) \\ + E\gamma_M \left(\nu\epsilon T + \sigma \sup_{0 \leq u \leq \epsilon T} w(u) + N(\epsilon T)\right), \end{aligned}$$

and

$$\begin{aligned} E\gamma_M \left(\nu\epsilon T + \sigma \sup_{0 \leq u \leq \epsilon T} w(u) + N(\epsilon T)\right) \leq \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} \gamma_M(\nu\epsilon T + \sigma\sqrt{\epsilon T}\xi) \exp -\frac{\xi^2}{2} d\xi \\ + \frac{2}{\sqrt{2\pi}} \lambda\epsilon T \int_0^{+\infty} \exp -\frac{\xi^2}{2} \left(\int_0^{+\infty} \gamma_M(\nu\epsilon T + \sigma\sqrt{\epsilon T}\xi + v)\mu(v)dv\right) d\xi. \end{aligned}$$

We can collect estimates and write

$$|\varphi_\epsilon(x) - \varphi(x)| \leq 2||\varphi|| \exp -T + C_M(\epsilon T),$$

and

$$C_M(\epsilon T) \to 0, \text{ as } \epsilon T \to 0.$$

The result (15.6.37) follows easily.

We now complete the proof of (15.6.26). We assume $\varphi(s) = \varphi(S)$. We have first, using this assumption and assuming

$$s + a_0 < S < M, \quad s + 2a_0 < M$$

(a_0 will be small so it is not a restriction), then applying (15.6.37)

$$\begin{aligned} &\sup_{s \leq x \leq M} |\varphi_\epsilon(x) - \varphi(x)| \\ &\leq \max\left[\sup_{s + a_0 \leq x \leq M} |\varphi_\epsilon(x) - \varphi(x)|, \sup_{s \leq x \leq s + a_0} |\varphi_\epsilon(x) - \varphi(x)|\right] \\ &\leq \sup_{s + a_0 \leq x \leq M} |\varphi_\epsilon(x) - \varphi(x)| + \sup_{s \leq x \leq s + a_0} |\varphi_\epsilon(x) - \varphi(s)| + \sup_{s \leq x \leq s + a_0} |\varphi(x) - \varphi(s)|. \end{aligned}$$

The first term follows from (15.6.37), the third term will be treated by the uniform continuity of $\varphi(x)$ on $[s,M]$. So we concentrate on the second term. We can write

$$\varphi_\epsilon(x)-\varphi(s)=\int_0^T \exp -u\ (E\varphi(y_x(\epsilon u)-\varphi(s))du$$
$$+\int_T^{+\infty}\exp -u\ E\varphi(y_x(\epsilon u)-\varphi(s)\exp -T.$$

Consider $u\le T$, then after considering possible cases, one can convince oneself of the relation

$$\begin{aligned}E\varphi(y_x(\epsilon u))-\varphi(s)&=-\varphi(s)[E1\!\!1_{\tau_s^x\le\epsilon u}1\!\!1_{|D(\epsilon u)-D(\tau_s^x)|\ge a_0}+E1\!\!1_{\tau_s^x>\epsilon u}1\!\!1_{|D(\epsilon u)|\ge a_0}]\\&+E(\varphi(x-D(\epsilon u))-\varphi(s))1\!\!1_{\tau_s^x>\epsilon u}1\!\!1_{|D(\epsilon u)|<a_0}\\&+E\varphi(S-D(\epsilon u)+D(\tau_s^x))1\!\!1_{\tau_s^x\le\epsilon u}1\!\!1_{|D(\epsilon u)-D(\tau_s^x)|\ge a_0}\\&+E(\varphi(S-D(\epsilon u)+D(\tau_s^x))-\varphi(S))1\!\!1_{\tau_s^x\le\epsilon u}1\!\!1_{|D(\epsilon u)-D(\tau_s^x)|<a_0}\\&+E\varphi(y_x(\epsilon u))1\!\!1_{\tau_{x,2}\le\epsilon u}.\end{aligned}$$

We then take account of the estimates

$$1\!\!1_{|D(\epsilon u)|\ge a_0}\le 1\!\!1_{\nu\epsilon T+N(\epsilon T)+\sigma\sup_{0\le u\le\epsilon T}w(u)}+1\!\!1_{\nu\epsilon T+N(\epsilon T)+\sigma\sup_{0\le u\le\epsilon T}-w(u)},$$

and therefore

$$E1\!\!1_{|D(\epsilon u)|\ge a_0}\le 2\Theta_{a_o}(\epsilon T).$$

With some additional effort we can also check that

$$E1\!\!1_{\tau_s^x\le\epsilon u}1\!\!1_{|D(\epsilon u)-D(\tau_s^x)|\ge a_0}\le 2\Theta_{a_o}(\epsilon T);$$
$$E1\!\!1_{\tau_{x,2}\le\epsilon u}\le\Theta_{S-s}(\epsilon T).$$

Also

$$E|\varphi(x-D(\epsilon u))-\varphi(s)|1\!\!1_{|D(\epsilon u)|<a_0}\le\gamma_M(2a_0);$$
$$E|\varphi(S-D(\epsilon u)+D(\tau_s^x))-\varphi(S)|1\!\!1_{|D(\epsilon u)-D(\tau_s^x)|<a_0}\le\gamma_M(a_0),$$

where $\gamma_M(\delta)$ is the modulus of continuity of $\varphi(x)$ on $[s,M]$. Collecting results we can assert that for $u\le T$

$$|E\varphi(y_x(\epsilon u))-\varphi(s)|\le||\varphi||(4\Theta_{a_o}(\epsilon T)+\Theta_{S-s}(\epsilon T))+\gamma_M(2a_0)+\gamma_M(a_0),$$

and thus

$$\sup_{s\le x\le s+a_0}|\varphi_\epsilon(x)-\varphi(s)|\le||\varphi||(4\Theta_{a_o}(\epsilon T)+\Theta_{S-s}(\epsilon T)+2\exp -T)+\gamma_M(2a_0)+\gamma_M(a_0),$$

which implies

$$\limsup_{\epsilon\to 0}\sup_{s\le x\le s+a_0}|\varphi_\epsilon(x)-\varphi(s)|\le 2||\varphi||\exp -T+\gamma_M(2a_0)+\gamma_M(a_0).$$

Therefore also

$$\limsup_{\epsilon\to 0}\sup_{s\le x\le M}|\varphi_\epsilon(x)-\varphi(x)|\le 2||\varphi||\exp -T+\gamma_M(2a_0)+\gamma_M(a_0)$$
$$+\sup_{s\le x\le s+a_0}|\varphi(x)-\varphi(s)|.$$

Letting a_0 tend to 0, then T tend to $+\infty$, we see that the right hand side is 0. This completes the proof of intermediary results and thus also the proof of Theorem 15.5 □

15.6.4. CHAPMAN-KOLMOGOROV EQUATION. From the formula (15.6.34) we can see that $y_x(t)$ has a density for any $t > 0$. This follows directly that the fact that $x - D(t)$ has a density for $t > 0$. Therefore for $\varphi(x) \geq 0$,

$$\Psi(x,t) \leq E\varphi(x - D(t)) = \int \varpi(\xi,t)\varphi(\xi)d\xi,$$

where $\varpi(\xi,t)$ is the density of $x - D(t)$. Therefore we can write

$$z(x,t) = E\varphi(y_x(t)) = \int p(\xi,t)\varphi(\xi)d\xi,$$

and $p(\xi,t)$ is the density of the variable $y_x(t)$. Formally we can write

$$p(\xi,0) = \delta(\xi - x).$$

Since $z(x,t) = 0$ if $\varphi = 0$ on $(s,+\infty)$ (see formula (15.6.34) again) we have $p(\xi,t) = 0$ for $\xi \leq s$. The Chapman-Kolmogorov equation describes the evolution of $p(\xi,t)$. With considerations similar as those for the invariant measure we can derive this equation and obtain

$$\begin{aligned}
&\frac{\partial p}{\partial t} - \tfrac{\sigma^2}{2}\frac{\partial^2 p}{\partial \xi^2} - \nu\frac{\partial p}{\partial \xi} - \lambda \textstyle\int_0^{+\infty}(p(\xi+\eta,t) - p(\xi,t))\mu(\eta)d\eta = 0, \quad && \text{if } \xi > s\\
&p(\xi,t) = 0, && \text{if } \xi \leq s\\
&p(\xi,0) = \delta(\xi - x)
\end{aligned} \tag{15.6.41}$$

We have proven the weak convergence

$$\int p(\xi,t)\varphi(\xi)d\xi \to \int m(\xi)\varphi(\xi)d\xi, \text{ as } \quad t \to +\infty.$$

Note that $\dfrac{\partial p}{\partial \xi}$ has a discontinuity in S with the relation

$$\frac{\sigma^2}{2}\left(\frac{\partial p}{\partial \xi}(S+0,t) - \frac{\partial p}{\partial \xi}(S-0,t) + \frac{\partial p}{\partial \xi}(s+0,t)\right) + \lambda\int_s^{+\infty} p(x,t)\int_{x-s}^{+\infty}\mu(\xi)d\xi = 0.$$

We do not develop details on the Chapman-Kolmogorov equation, since it has no explicit solution, and all results on ergodic theory can bypass this equation.

15.6.5. COST ASSOCIATED WITH AN s, S POLICY. We can now compute the cost associated with an s,S policy as defined by (17.2.2) for a general impulse control. The impulse times are the random times $\tau_{x,n}^{s,S}$ and the impulses (orders) are $S - y_x^{s,S}(\tau_{x,n}^{s,S} - 0), \quad n \geq 1$. Recalling formula (15.6.8), and using similar reasoning one convinces oneself that the associated cost is given by

$$\begin{aligned}
u_\alpha^{s,S}(x) = {} & E\int_0^{\tau_s^x} \exp -\alpha t\, f(y_x^{s,S}(t))dt + E[\exp -\alpha\tau_s^x\ (K\\
& +c(S - y_x^{s,S}(\tau_s^x - 0)))] + E\exp -\alpha\tau_s^x\ \frac{E\int_0^{\tau_s^S}\exp -\alpha t\ f(y_S^{s,S}(t))dt}{1 - E\exp -\alpha\tau_s^S}\\
& +E\exp -\alpha\tau_s^x\ \frac{E\left[\exp -\alpha\tau_s^S\ (K + c(S - y_S^{s,S}(\tau_s^S - 0)))\right]}{1 - E\exp -\alpha\tau_s^S}.
\end{aligned} \tag{15.6.42}$$

We can also write this formula as

$$(15.6.43)\quad u_\alpha^{s,S}(x) = E\int_0^{\tau_s^x} \exp -\alpha t\ f(y_x^{s,S}(t))dt + cE[\exp -\alpha\tau_s^x\ (s -y_x^{s,S}(\tau_s^x - 0))] + \frac{E\exp -\alpha\tau_s^x}{1-E\exp -\alpha\tau_s^S}\left[E\int_0^{\tau_s^S} \exp -\alpha t\ f(y_S^{s,S}(t))dt +K + c(S-s) + cE[\exp -\alpha\tau_s^S(s - y_S^{s,S}(\tau_s^S - 0))\right],$$

and thus

$$(15.6.44)\quad u_\alpha^{s,S}(S) = -(K + c(S-s)) + \frac{1}{1-E\exp -\alpha\tau_s^S}\left[E\int_0^{\tau_s^S} \exp -\alpha t\ f(y_S^{s,S}(t))dt + K +c(S-s) + cE[\exp -\alpha\tau_s^S(s - y_S^{s,S}(\tau_s^S - 0))\right].$$

So we may write

$$(15.6.45)\quad u_\alpha^{s,S}(x) = E\int_0^{\tau_s^x} \exp -\alpha t\ f(y_x^{s,S}(t))dt + cE[\exp -\alpha\tau_s^x\ (s -y_x^{s,S}(\tau_s^x - 0))] + E\exp -\alpha\tau_s^x\ (u_\alpha^{s,S}(S) + K + c(S-s)).$$

If we introduce

$$G_\alpha^{s,S}(x) = u_\alpha^{s,S}(x) + cx,$$

then we get

$$\begin{aligned} G_\alpha^{s,S}(x) &= E\int_0^{\tau_s^x} \exp -\alpha t\ f(y_x^{s,S}(t))dt + c(x - E[\exp -\alpha\tau_s^x\ y_x^{s,S}(\tau_s^x - 0)]) \\ &+ E\exp -\alpha\tau_s^x\ (K + G_\alpha^{s,S}(S)). \end{aligned}$$

We use the relation

$$x - E[\exp -\alpha\tau_s^x\ y_x^{s,S}(\tau_s^x - 0)] = E\int_0^{\tau_s^x} \exp -\alpha t\ (\nu + \lambda\bar{\xi} + \alpha y_x^{s,S}(t))dt,$$

which implies

$$(15.6.46)\quad G_\alpha^{s,S}(x) = E\int_0^{\tau_s^x} \exp -\alpha t\ g_\alpha(y_x^{s,S}(t))dt + E\exp -\alpha\tau_s^x\ (K + G_\alpha^{s,S}(S)),$$

in which

$$g_\alpha(x) = f(x) + c\nu + c\lambda\bar{\xi} + \alpha cx;$$

$$(15.6.47)\quad G_\alpha^{s,S}(S) = \frac{E\int_0^{\tau_s^S} \exp -\alpha t\ g_\alpha(y_S^{s,S}(t))dt + KE\exp -\alpha\tau_s^S}{1 - E\exp -\alpha\tau_s^S},$$

and we recover that $G_\alpha^{s,S}(x)$ is solution of the problem

$$(15.6.48)\quad \begin{aligned} AG_\alpha^{s,S}(x) + \alpha G_\alpha^{s,S}(x) &= g_\alpha(x) \quad \forall x > s \\ G_\alpha^{s,S}(x) &= K + G_\alpha^{s,S}(S) \quad \forall x \le s \end{aligned}$$

15.6.6. OPTIMIZATION. Can we optimize among s, S values? We know from the Q.V.I. approach that there exists a unique pair s, S which minimizes $G_\alpha^{s,S}(x)$ for any x. We recall that it is the solution of the following algebraic system. Set

$$H_\alpha^{s,S}(x) = (G_\alpha^{s,S})'(x),$$

which is not continuous for $x = s$. The conditions are

$$H_\alpha^{s,S}(s) = 0 \quad H_\alpha^{s,S}(S) = 0, \tag{15.6.49}$$

and the corresponding function becomes C^1. The second condition is clear from the general formula (15.6.26). Indeed S enters only through the term $G_\alpha^{s,S}(S)$ which should be minimized. Note indeed that in the integral the part of the process $y_S^{s,S}(t)$ before the stopping time τ_s^x does not depend on S.

The second condition is less obvious. Finding a unique s which minimizes $G_\alpha^{s,S}(x)$ for any x does not seem obvious from formula (15.6.26). In particular it should minimizes $G_\alpha^{s,S}(S)$. We then state the

Proposition 15.3. *There exists a pair s, S which minimizes $G_\alpha^{s,S}(S)$. It satisfies conditions* (15.6.29) *and minimizes $G_\alpha^{s,S}(x)$ for any x.*

PROOF. We will proceed with direct checking, since we have explicit formulas.
We note from (15.6.27) that

$$G_\alpha^{s,S}(S) = -K + \frac{E\int_0^{\tau_s^S} \exp -\alpha t \; g_\alpha(y_S^{s,S}(t))dt + K}{1 - E\exp -\alpha\tau_s^S},$$

so we have to minimize the expression

$$G_\alpha^{s,S}(s) = \frac{E\int_0^{\tau_s^S} \exp -\alpha t \; g_\alpha(y_S^{s,S}(t))dt + K}{1 - E\exp -\alpha\tau_s^S}.$$

We then recall that

$$z_{\alpha,s}(x) = E\int_0^{\tau_s^x} \exp -\alpha t \; g_\alpha(y_x(t))dt,$$

where $y_x(t) = x - D(t)$ is the solution of

$$\begin{aligned} Az_{\alpha,s}(x) + \alpha z_{\alpha,s}(x) &= g_\alpha(x), \quad && x > s \\ z_{\alpha,s}(x) &= 0, && x \le s \end{aligned}$$

and is given by the formula

$$z_{\alpha,s}(x) = \frac{2}{\sigma^2}\int_s^x \Gamma_\alpha(x-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g_\alpha(\eta)d\eta\right)d\xi,$$

and

$$E\int_0^{\tau_s^S} \exp -\alpha t \; g_\alpha(y_S^{s,S}(t))dt = z_{\alpha,s}(S).$$

After rearranging the quantity to optimize becomes

$$\Theta_\alpha(s,S) = \frac{\int_s^S \Gamma_\alpha(S-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g_\alpha(\eta)d\eta\right)d\xi + K\frac{\sigma^2}{2}}{\int_0^{S-s}\Gamma_\alpha(x)dx}.$$

The algebraic equations coming from the Q.V.I., recalling the expression of $H_{\alpha,s}(x)$, see (15.5.5) and (15.5.6) amount to

$$\int_s^S \Gamma_\alpha(S-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g'_\alpha(\eta)d\eta\right)d\xi = 0, \tag{15.6.50}$$

and

$$\int_s^S \left[\int_s^x \Gamma_\alpha(x-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g'_\alpha(\eta)d\eta\right)d\xi\right]dx + K\frac{\sigma^2}{2} = 0. \tag{15.6.51}$$

So what we have to do boils down to checking that in equaling to 0 the derivatives of $\Theta_\alpha(s,S)$ with respect to s and S we obtain indeed (15.6.30) and (15.6.31). This is a pure algebraic equivalence. It is first useful to check that

$$\int_s^x \Gamma_\alpha(x-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g'_\alpha(\eta)d\eta\right)d\xi - \int_s^x \Gamma_\alpha(x-\xi)g_\alpha(\xi)d\xi$$
$$-\beta_{2\alpha}\int_s^x \Gamma_\alpha(x-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g_\alpha(\eta)d\eta\right)d\xi.$$

Therefore (15.6.30) and (15.6.31) become

$$\int_s^S \Gamma_\alpha(S-\xi)g_\alpha(\xi)d\xi + \beta_{2\alpha}\int_s^S \Gamma_\alpha(S-\xi) \cdot\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g_\alpha(\eta)d\eta\right)d\xi = 0, \tag{15.6.52}$$

and

$$K\frac{\sigma^2}{2} - \int_s^S \left(\int_s^x \Gamma_\alpha(x-\xi)g_\alpha(\xi)d\xi\right)dx - \beta_{2\alpha} \cdot\int_s^S \left[\int_s^x \Gamma_\alpha(x-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g_\alpha(\eta)d\eta\right)d\xi\right]dx = 0. \tag{15.6.53}$$

But this last relation is equivalent to

$$K\frac{\sigma^2}{2} - \int_s^{+\infty} \exp\beta_{2\alpha}(\eta-s)g_\alpha(\eta)d\eta\int_0^{S-s}\Gamma_\alpha(y)dy + \int_s^S \Gamma_\alpha(S-\xi)\left(\int_\xi^{+\infty} \exp\beta_{2\alpha}(\eta-\xi)g_\alpha(\eta)d\eta\right)d\xi = 0. \tag{15.6.54}$$

If we now take the derivatives of $\Theta_\alpha(s,S)$ with respect to s and S, we can check easily after rearrangement and combining equations that we get indeed the relations (15.6.52) and (15.6.54).

Since we recover the values which allow to solve the Q.V.I. we know that the value of s is uniquely defined, and S is also uniquely defined, provided we take the smallest minimum. Since we have solved the Q.V.I. $G_\alpha^{s,S}(x)$ is minimized for any x. This concludes the proof. □

15.6.7. ERGODIC CASE. We can now easily treat the ergodic case, as $\alpha \to 0$. We can check immediately that

$$\lim_{\alpha\to 0}\alpha u_\alpha^{s,S}(x) = \lim_{\alpha\to 0}\alpha G_\alpha^{s,S}(x) = \lim_{\alpha\to 0}\alpha G_\alpha^{s,S}(s) = c\nu + c\lambda\bar{\xi} + \frac{K + E\int_0^{\tau_s^S} f(y_S(t))dt}{E\tau_s^S} = \rho^{s,S}. \tag{15.6.55}$$

Moreover $G_\alpha^{s,S}(x) - G_\alpha^{s,S}(s) \to G^{s,S}(x)$ uniformly on compact sets and $G^{s,S}(x)$ is the solution of

$$\begin{aligned} AG^{s,S}(x) + \rho^{s,S} &= c\nu + c\lambda\bar{\xi} + f(x), \quad && x > s \\ G^{s,S}(x) &= 0, && x \leq s \\ 0 &= K + G^{s,S}(S) \end{aligned} \tag{15.6.56}$$

From this equation we can verify that

$$\begin{aligned} \rho^{s,S} &= c\nu + c\lambda\bar{\xi} + \int_s^{+\infty} f(x)m^{s,S}(x)dx \\ &+K\frac{\sigma^2}{2}((m^{s,S})'(S-0) - (m^{s,S})'(S+0)), \end{aligned} \tag{15.6.57}$$

which is also expression (15.6.55). Minimizing $\rho^{s,S}$ with respect to s, S leads to conditions

$$(G^{s,S})'(S) = 0 \quad (G^{s,S})'(s) = 0,$$

and we recover the Q.V.I. of the ergodic case, see section 15.5.

CHAPTER 16

MEAN-REVERTING INVENTORY CONTROL

16.1. INTRODUCTION

In this chapter, we consider a situation in which the control of inventories is done partly through impulse controls as in the previous chapter, and partly through a smoothing procedure called mean-reverting. The mean-reverting procedure is a given continuous feedback, which continuously increases or decreases the inventory when it is below or above a given level. This action is not optimized and its cost being fixed is omitted. The impulse control, which remains the decision, is thus a complement. We will check that the s, S policy remains valid, with naturally different values of the quantities s, S. The demand is a continuous diffusion, without jumps. The case of jumps raises mathematical difficulties. The mean-reverting inventory model has been considered in [**13**] and [**14**], with different techniques. In fact, in these papers a two-band impulse control problem is considered. Unexpectedly, this situation turns out to be simpler, because one needs to solve differential equations in a bounded domain, instead of differential equations in unbounded domains as in the present case. There is also a difference in the cost function with these papers, in which a quadratic cost is considered, whereas here our traditional holding and shortage costs are considered. The quadratic cost is justified by the fact that one tries to remain close to the targeted inventory and thus one minimizes the distance to the target.

16.2. DESCRIPTION OF THE PROBLEM

16.2.1. THE MODEL. The demand process is composed of two parts: first a deterministic part with constant rate ν. We next consider a probability space $\Omega, \mathcal{A}, P$ on which is defined a Wiener process $w(t)$.

The demand on an interval $(0, t)$ is then given by

$$D(t) = \nu t + \sigma w(t),$$

where σ is a positive coefficient. Let $\mathcal{F}^t = \sigma(D(s), s \leq t)$

An impulse control is a sequence

$$\theta_n, v_n,$$

where θ_n is a stopping time with respect to the filtration $\mathcal{F}^t$ and v_n is a random variable $\mathcal{F}^{\theta_n}$ measurable. Denoting by V an impulse control, the corresponding inventory is described by the formula

$$y_x(t; V) = x + k\left(\gamma t - \int_0^t y_x(s; V)ds\right) - D(t) + M(t; V), \tag{16.2.1}$$

with

$$M(t) = M(t;V) = \sum_{\{n|\theta_n<t\}} v_n.$$

The new element in equation (17.2.1) is the term

$$k\left(\gamma t - \int_0^t y_x(s;V)ds\right),$$

which describes the mean-reverting procedure. The increase or decrease of the inventory takes place in a continuous manner. Therefore during an interval of time dt, we increase the inventory by an amount $k(\gamma - y(t))dt$ if the inventory $y(t)$ is smaller that the target $\gamma \geq 0$, $k \geq 0$. It is alternatively decreased by the same amount if the inventory is larger than γ. If $k = 0$ the mean-reverting part disappears and we recover the model of the previous chapter, except that there are no jumps. Note that the term $k\gamma t$ can be combined with the deterministic part of the demand $-\nu t$, and this will be apparent in the mathematical analysis. However, these terms have a different economic interpretation, so we do not combine them. Accordingly, we do not make any assumption on the sign of $k\gamma - \nu$.

The cost functional will be identical to the situation without the mean-reverting part. let

$$f(x) = hx^+ + px^-.$$

We define the cost functional by

$$J_x(V) = E\left[\sum_{n=0}^{\infty}(K + cv_n)\exp -\alpha\theta_n + \int_0^{\infty} f(y_x(t;V))\exp -\alpha t dt\right], \tag{16.2.2}$$

and the value function

$$u(x) = \inf\ J_x(V).$$

16.2.2. Q.V.I. We can then write the Q.V.I arising from Dynamic Programming. We introduce the operator

$$Au(x) = -\frac{1}{2}\sigma^2 u''(x) + (\nu + k(x-\gamma))u'(x), \tag{16.2.3}$$

in which we see the term originated from the mean-reverting part. The operator A has not constant coefficients anymore.

The Q.V.I. is given by

$$\begin{gathered} Au(x) + \alpha u(x) \leq f(x) \\ u(x) \leq M(u)(x) \\ (Au(x) + \alpha u(x) - f(x))(u(x) - M(u)(x)) = 0 \end{gathered} \tag{16.2.4}$$

with

$$M(u)(x) = K + \inf_{v>0}[cv + u(x+v)].$$

As usual we look for a function which is C^1 with linear growth. Standard arguments show that the solution of the Q.V.I is the value function.

16.2.3. TRANSFORMATION. As done in the previous chapter, it is convenient to consider the transformation

$$G(x) = u(x) + cx,$$

and thus the problem becomes: find a function $G(x)$, with $x \in R$, which is C^1 and has linear growth, satisfying

$$\begin{gathered} AG(x) + \alpha G(x) \leq g(x) + c(\nu - k\gamma) \\ G(x) \leq K + \inf_{\eta \geq x} G(\eta) \\ (AG(x) + \alpha G(x) - g(x) - c(\nu - k\gamma))(G(x) - K - \inf_{\eta \geq x} G(\eta)) = 0 \end{gathered} \tag{16.2.5}$$

The function g is given by

$$g(x) = f(x) + c(\alpha + k)x. \tag{16.2.6}$$

16.3. s, S POLICY

16.3.1. THRESHOLD s. Given a number s, we look for a solution of (16.2.5) as follows:

$$\begin{aligned} AG_s(x) + \alpha G_s(x) &= g(x) + c(\nu - k\gamma), && x > s \\ G_s(x) &= K + \inf_{\eta \geq s} G_s(\eta), && x \leq s \end{aligned} \tag{16.3.1}$$

In order to get a function which is globally C^1 we impose the condition

$$G_s'(s) = 0. \tag{16.3.2}$$

This is a relation to define s. We proceed by considering

$$H_s(x) = G_s'(x),$$

which is the solution of

$$\begin{aligned} AH_s(x) + (\alpha + k)H_s(x) &= g'(x), && x > s \\ H_s(x) &= 0, && x \leq s \end{aligned} \tag{16.3.3}$$

we look for a bounded continuous solution $H_s(x)$ of the Dirichlet problem (16.3.3), for a given s. To define s, we use the second relation (16.3.1) which can be written as follows

$$0 = K + \inf_{\eta \geq s} \int_s^\eta H_s(\eta) d\eta. \tag{16.3.4}$$

Note that

$$g'(x) = \begin{cases} h + c(\alpha + k), & \text{if } x > 0 \\ -p + c(\alpha + k), & \text{if } x < 0 \end{cases} \tag{16.3.5}$$

We make the assumption

$$-p + c(\alpha + k) < 0. \tag{16.3.6}$$

16.3.2. GREEN FUNCTION. Although it is possible to prove the existence of a solution of (16.3.3) by an approximation procedure, it is important to obtain an explicit formula as we have done in the standard case. For that purpose we need a Green function, solution of the differential equation in the whole space, without right hand side. We introduce the following problem

$$-\tfrac{1}{2}\sigma^2\Phi"(x) + (\nu + k(x-\gamma))\Phi'(x) + (\alpha+k)\Phi(x) = 0, \quad \forall x \in R \tag{16.3.7}$$
$$\Phi(0) = 1, \quad \Phi(+\infty) = 0$$

We want to prove the

Theorem 16.1. *We assume* (16.3.6). *There exists one and only one solution of* (16.3.7) *such that*

$$\Phi(x) > 0,\ \Phi'(x) < 0,\ \Phi"(x) > 0. \tag{16.3.8}$$

As $x \to -\infty, \Phi(x) \to +\infty,\ \Phi'(x) \to -\infty$. *Considering the roots* $\beta_k < 0,\ \beta_k^* > 0$ *of*

$$-\frac{1}{2}\sigma^2\beta^2 + \nu\beta + \alpha + k = 0,$$

we have the growth condition

$$0 > \Phi'(x) \geq \frac{\exp\dfrac{k}{\sigma^2}(x-\gamma)^2}{\beta_k^* - \beta_k}\left[\beta_k\left(\beta_k^* - \Phi'(0)\exp -\frac{k\gamma^2}{\sigma^2}\right)\exp\beta_k x \right. \tag{16.3.9}$$
$$\left. +\beta_k^*\left(\Phi'(0)\exp -\frac{k\gamma^2}{\sigma^2} - \beta_k\right)\exp\beta_k^* x\right], \quad x \leq 0$$

Proof. We begin by solving (16.3.7) in $[0,+\infty)$. It is a Dirichlet type boundary value problem. It has one and only one solution and

$$0 \leq \Phi(x) \leq 1,\ x \geq 0,$$

and we have the energy relation

$$\frac{\sigma^2}{2}\int_0^{+\infty}\Phi'^2 dx + \left(\alpha + \frac{k}{2}\right)\int_0^{+\infty}\Phi^2 dx = -\frac{\sigma^2}{2}\Phi'(0) + \frac{1}{2}(\nu - k\gamma). \tag{16.3.10}$$

If we next test the equation (16.3.7) with $\Phi'(x)$ and integrate between 0 and $x > 0$. We obtain

$$-\frac{\sigma^2}{4}\Phi'^2(x) + \frac{\sigma^2}{4}\Phi'^2(0) + \int_0^x (\nu + k(\xi-\gamma))\Phi'^2(\xi)d\xi + \frac{\alpha+k}{2}(\Phi^2(x) - 1) = 0,$$

From this formula, one can convince oneself easily that $\Phi'^2(x)$ has a limit as $x \to +\infty$, but since $\Phi'^2(x)$ is also an L^1 function, necessarily this limit is 0. Therefore, we obtain an additional energy relation

$$\frac{\sigma^2}{4}\Phi'^2(0) + \int_0^{+\infty}(\nu + k(x-\gamma))\Phi'^2(x)dx = \frac{\alpha+k}{2}. \tag{16.3.11}$$

Since $\Phi(\epsilon) < 1 = \Phi(0)$, we see that $\Phi'(0) \leq 0$. In fact $\Phi'(0) < 0$. Indeed, if we set $\Psi(x) = \Phi'(x)$, then $\Psi(x)$ satisfies

$$-\frac{1}{2}\sigma^2\Psi"(x) + (\nu + k(x-\gamma))\Psi'(x) + (\alpha+2k)\Psi(x) = 0,\ x > 0, \tag{16.3.12}$$

and we know already that $\Psi(+\infty) = 0$. Therefore $\Psi(0)$ cannot be 0, otherwise $\Psi(x)$ would be identically 0, hence $\Phi(x)$ would be constant on $[0,\infty)$ which is impossible. Since $\Psi(0) \leq 0$, in fact $\Psi(0) < 0$. Necessarily $\Psi(x) \leq 0$ and cannot be 0 at any point, otherwise this point would be a maximum, which is impossible. Therefore $\Psi(x) < 0, \ \forall x \geq 0$. In addition $\Psi'(x)$ cannot be negative or zero at any point. Otherwise, $\Psi(x)$ would have a negative local minimum, which is also impossible. Therefore $\Psi'(x) = \Phi"(x) > 0, \ \forall x \in [0, \infty)$.

We thus have proved the properties (16.3.8) for $x \in [0, \infty)$.

We next solve equation (16.3.7) in $(-\infty, 0]$ as a two-dimensional system of linear differential equations, with given initial values at 0. For convenience, we will change the sign of x, to solve a system on $[0, \infty)$. Define

$$y(x) = \Phi(-x), \ z(x) = \Phi'(-x), \ x > 0$$

then the pair $y(x)$, $z(x)$ is solution of

$$\begin{aligned} y' &= -z; \\ z' &= \tfrac{2}{\sigma^2}((kx - (\nu - k\gamma))z - (\alpha + k)y), \end{aligned} \tag{16.3.13}$$

on $[0, \infty)$ with initial conditions $y(0) = 1, \ z(0) = z_0 = \Phi'(0) < 0$. Consider the matrix

$$\mathcal{A}(x) = \begin{pmatrix} 0 & -1 \\ -\frac{2}{\sigma^2}(\alpha + k) & \frac{2}{\sigma^2}(kx - (\nu - k\gamma) \end{pmatrix},$$

then we associate to it a family of matrices, called $\Sigma(x, x_0), \ x \geq x_0$, the *fundamental matrix* satisfying the differential equation

$$\begin{aligned} \frac{d}{dx}\Sigma(x, x_0) &= \mathcal{A}(x)\Sigma(x, x_0); \\ \Sigma(x_0, x_0) &= I, \end{aligned}$$

where I is the identity. From the theory of linear differential equations, the matrix $\Sigma(x, x_0)$ is well defined, for any pair $x \geq x_0$. Moreover the solution of (16.3.13) is given by

$$\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \Sigma(x, 0) \begin{pmatrix} 1 \\ z_0 \end{pmatrix},$$

so it is uniquely defined for any $x > 0$. Let

$$\varpi(x) = \exp -\frac{2}{\sigma^2}\left[k\frac{(x + \gamma)^2}{2} - \nu x\right],$$

and

$$\tilde{z}(x) = z(x)\varpi(x),$$

then the pair $y(x), \tilde{z}(x)$ is the solution of

$$\begin{aligned} y' &= -\frac{\tilde{z}}{\varpi} \\ \tilde{z}' &= -\frac{2}{\sigma^2}(\alpha + k)\varpi y \end{aligned} \tag{16.3.14}$$

with initial conditions $y(0) = 1, \ \tilde{z}(0) = z_0 \exp -\dfrac{k\gamma^2}{\sigma^2}$. It follows easily that $y(x) > 0, \ \tilde{z}(x) < 0$. After easy manipulations, we get the following integral equation for $\tilde{z}(x)$

$$\begin{aligned} \tilde{z}(x) = \tilde{z}(0) - \frac{2}{\sigma^2}(\alpha + k)y(0)\int_0^x \varpi(\xi)d\xi \\ + \frac{2}{\sigma^2}(\alpha + k)\int_0^x G(x, \eta)\tilde{z}(\eta)d\eta, \end{aligned} \tag{16.3.15}$$

with

$$G(x,\eta)=\int_\eta^x \frac{\varpi(\xi)}{\varpi(\eta)}d\xi,$$

and

$$\frac{\varpi(\xi)}{\varpi(\eta)}=\exp -\frac{2}{\sigma^2}\left[\frac{k}{2}(\xi-\eta)(\xi+\eta+2\gamma)-\nu(\xi-\eta)\right].$$

For $\xi\geq\eta\geq 0$, we have

$$\frac{\varpi(\xi)}{\varpi(\eta)}\leq \exp\frac{2\nu}{\sigma^2}(\xi-\eta),$$

hence for $x\geq\eta\geq 0$

$$G(x,\eta)\leq\frac{\exp\dfrac{2\nu}{\sigma^2}(x-\eta)-1}{\dfrac{2\nu}{\sigma^2}}.$$

Also

$$\varpi(x)\leq\exp\frac{2\nu}{\sigma^2}x.$$

We use these inequalities in (16.3.15) and note that $y(0)=1$ and $\tilde{z}(\eta)\leq 0$. After rearrangements we obtain the inequality

$$\begin{aligned}(16.3.16)\quad \tilde{z}(x)\geq\tilde{z}(0)&+\frac{\alpha+k}{\nu}\,y(0)-\frac{\alpha+k}{\nu}\\ &\cdot\exp\frac{2\nu}{\sigma^2}x\,\left(y(0)-\int_0^x\tilde{z}(\eta)\exp-\frac{2\nu}{\sigma^2}\eta d\eta\right)-\frac{\alpha+k}{\nu}\int_0^x\tilde{z}(\eta)d\eta.\end{aligned}$$

We then introduce the function $u(x)$ solution of

$$\begin{aligned}(16.3.17)\quad u(x)=\tilde{z}(0)&+\frac{\alpha+k}{\nu}\,y(0)-\frac{\alpha+k}{\nu}\\ &\cdot\exp\frac{2\nu}{\sigma^2}x\,\left(y(0)-\int_0^x u(\eta)\exp-\frac{2\nu}{\sigma^2}\eta d\eta\right)-\frac{\alpha+k}{\nu}\int_0^x u(\eta)d\eta.\end{aligned}$$

One checks easily that $u(x)$ satisfies

$$-\frac{\sigma^2}{2}u''(x)+\nu u'(x)+(\alpha+k)u(x)=0,$$

and we also have

$$u(0)=\tilde{z}(0),\quad u'(0)=-\frac{2}{\sigma^2}(\alpha+k)y(0).$$

Considering the roots of

$$-\frac{\sigma^2}{2}\beta^2+\nu\beta+(\alpha+k)=0,$$

denoted by $\beta_k<0$, $\beta_k^*>0$, we can express the function $u(x)$ by the formula

$$\begin{aligned}(16.3.18)\qquad u(x)=&\frac{\tilde{z}(0)}{\beta_k^*-\beta_k}(-\beta_k\exp\beta_k^*x+\beta_k^*\exp\beta_kx)\\ &-\frac{2}{\sigma^2}\frac{(\alpha+k)y(0)}{\beta_k^*-\beta_k}(\exp\beta_k^*x-\exp\beta_kx).\end{aligned}$$

Let us check that $\tilde{z}(x)\geq u(x)$. Indeed, if we set $w(x)=\tilde{z}(x)-u(x)$, then we have the inequity

$$w(x)\geq\frac{\alpha+k}{\nu}\int_0^x\left(\exp\frac{2\nu}{\sigma^2}(x-\eta)-1\right)w(\eta)d\eta,$$

hence

$$w(x) \geq -\frac{\alpha+k}{\nu}\int_0^x \left(\exp\frac{2\nu}{\sigma^2}(x-\eta)-1\right) w^-(\eta)d\eta;$$

$$w(x)^- \exp -\frac{2\nu}{\sigma^2}x \leq \frac{\alpha+k}{\nu}\int_0^x \exp -\frac{2\nu}{\sigma^2}\eta\ w^-(\eta)d\eta.$$

From Gronwall's inequity, we then assert that

$$w(x)^- \exp -\frac{2\nu}{\sigma^2}x = 0,$$

which implies the result. Recalling that

$$\tilde{z}(0) = -y'(0)\exp -\frac{k\gamma^2}{\sigma^2} = \Phi'(0)\exp -\frac{k\gamma^2}{\sigma^2};$$

$$y(0) = \Phi(0) = 1; \quad \beta_k^*\beta_k = -\frac{2}{\sigma^2}(\alpha+k); \quad \beta_k^* + \beta_k = \frac{2\nu}{\sigma^2},$$

and rearranging we can write

$$z(x)\exp -\frac{k}{\sigma^2}(x+\gamma)^2 \geq \beta_k \exp -\beta_k x \frac{\beta_k^* - \Phi'(0)\exp -\frac{k\gamma^2}{\sigma^2}}{\beta_k^*-\beta_k}$$

$$+\beta_k^* \exp -\beta_k^* x \ \frac{-\beta_k + \Phi'(0)\exp -\frac{k\gamma^2}{\sigma^2}}{\beta_k^*-\beta_k}.$$

Finally recalling that $z(x) = \Phi'(-x), x \geq 0$, we deduce easily the estimate (16.3.9). We have $\Phi(x) > 0,\ \Phi'(x) < 0$. Let s check that

$$\Phi'(x) \to -\infty, \text{ as } x \to -\infty. \tag{16.3.19}$$

Indeed, from (16.3.14) $\tilde{z}(x)$ is strictly decreasing. So

$$\begin{aligned} z(x) &= \tilde{z}(x)\exp\frac{2}{\sigma^2}\left[k\frac{(x+\gamma)^2}{2} - \nu x\right] \\ &\leq \tilde{z}(0)\exp\frac{2}{\sigma^2}\left[k\frac{(x+\gamma)^2}{2} - \nu x\right] \to -\infty, \text{as } x \to +\infty \end{aligned}$$

hence (16.3.19). Consider $\Psi(x) = \Phi'(x)$. It is a solution of (16.3.12) and $\Psi(x) < 0, \forall x \leq 0$. Moreover

$$\Psi(x) \to -\infty, \text{as } x \to -\infty$$

We also know that $\Psi'(0) = \Phi"(0) > 0$. Necessarily

$$\Psi'(x) = \Phi"(x) > 0, \forall x < 0$$

Otherwise there will be a point x_0 in which $\Psi'(x_0) = 0$. We can take the largest negative such point. It is a local minimum for $\Psi(x)$. But from equation (16.3.12) $\Psi(x)$ cannot have a negative local minimum for $x < 0$. This completes the proof. □

We call $\Phi(x)$ a Green function for the problem (16.3.3), by analogy with the solution of linear P.D.E. Note that when $k = 0$,

$$\Phi(x) = \exp\beta x,$$

where β is the negative root of

$$-\frac{\sigma^2}{2}\beta^2 + \nu\beta + \alpha = 0.$$

16.3.3. SOLUTION OF (16.3.3)**.** Thanks to the Green function we can give an explicit formula for the solution of (16.3.3). Let us set

$$\begin{aligned}\vartheta(x) &= \varpi(-x) \\ &= \exp -\frac{2}{\sigma^2}\left[k\frac{(x-\gamma)^2}{2}+\nu x\right].\end{aligned} \tag{16.3.20}$$

We begin by considering the function

$$\chi(x) = \vartheta(x)\Phi(x),\ x \in R, \tag{16.3.21}$$

which satisfies the differential equation

$$-\frac{\sigma^2}{2}\chi"(x) - (k(x-\gamma)+\nu)\chi'(x) + \alpha\chi(x) = 0 \tag{16.3.22}$$

$$\chi(0) = \exp -\frac{k}{\sigma^2}\gamma^2, \quad \chi(+\infty) = 0$$

Lemma 16.1. *There exists one and only one solution of* (16.3.22)*. Moreover*

$$\chi'(x) < 0, \quad \chi(x) \to +\infty,\ \textit{as}\ x \to -\infty. \tag{16.3.23}$$

PROOF. The function defined by (16.3.22) is a solution, by direct checking. The uniqueness follows from the theory of linear differential equations, as for $\Phi(x)$. We only have to prove that $\chi'(x) < 0$. We necessarily have $\chi'(0) < 0$, otherwise $\chi(x)$ would increase for $x > 0$ close to 0. But then, since it tends to 0 as $x \to +\infty$ $\chi(x)$ would have a positive maximum on $(0, +\infty)$ which is impossible. We then have

$$\chi'(x) = \vartheta(x)\Phi'(x) - \frac{2}{\sigma^2}(k(x-\gamma)+\nu).$$

Since $\Phi'(x) < 0$, we have $\chi'(x) < 0$ for $x \geq \gamma$. If $\chi'(x)$ vanishes at some point x^* in $(0, \gamma)$ then the first such point must be a local minimum. But then $\chi(x)$ would increase after x^*, close to x^*. But then there will be a positive local maximum, which is not possible. Hence $\chi'(x) < 0$, for $x \geq 0$. When $x < 0$, again $\chi'(x)$ cannot vanish and change sign otherwise $\chi(x)$ will have a positive local maximum. It cannot have an inflection point, since the second derivative would be 0, which is not possible from the second order equation (16.3.22). The fact that $\chi(x) \to +\infty$ as $x \to -\infty$ follows immediately from the fact that $\chi'(x) < 0$ and $\chi"(x) > 0$ for x sufficiently negative. □

Then we state the

Theorem 16.2. *The solution of* (16.3.3) *is unique and given by the formula*

$$H_s(x) = \frac{2}{\sigma^2}\int_s^x \frac{\Phi(x)}{\Phi(\xi)}\left(\int_\xi^{+\infty} g'(\eta)\frac{\chi(\eta)}{\chi(\xi)}d\eta\right)d\xi, \quad x \geq s. \tag{16.3.24}$$

PROOF. The first thing to prove is that the function defined by (16.3.24), called $H_s(x)$ for simplicity is bounded. Since $g'(x)$ is bounded, it is sufficient to prove that the function

$$Z_s(x) = \frac{2}{\sigma^2}\int_s^x \frac{\Phi(x)}{\Phi(\xi)}\left(\int_\xi^{+\infty} \frac{\chi(\eta)}{\chi(\xi)}d\eta\right)d\xi, \quad x \geq s$$

is bounded. From the properties of the functions $\Phi(x)$ and $\chi(x)$ we can check easily that it is sufficient to show that $Z_0(x)$ is bounded. But we are going to show that

$$Z_0(x) = \frac{1-\Phi(x)}{\alpha+k}. \tag{16.3.25}$$

Indeed, let us define by $Z_0(x)$ by (16.3.25). It is a bounded solution of

$$-\frac{\sigma^2}{2} Z_0" + (\nu + k(x-\gamma))Z_0' + (\alpha+k)Z_0 = 1, \quad x > 0 \tag{16.3.26}$$
$$Z_0(0) = 0$$

Define next

$$A(x) = \frac{Z_0(x)}{\Phi(x)} = -\frac{1}{\alpha+k} + \frac{1}{\alpha+k}\frac{1}{\Phi(x)}$$

then, by direct computation we have

$$A"(x) + A'(x)\left[2\frac{\Phi'(x)}{\Phi(x)} - \frac{2}{\sigma^2}(\nu + k(x-\gamma))\right] = -\frac{2}{\sigma^2\Phi(x)},$$

hence also

$$\frac{d}{dx}(A'(x)\Phi^2(x)\vartheta(x)) = -\frac{2}{\sigma^2}\Phi(x)\vartheta(x).$$

But also from the definition of $A(x)$

$$A'(x)\Phi^2(x)\vartheta(x) = -\frac{1}{\alpha+k}\Phi'(x)\vartheta(x).$$

Since on $x > 0$, $\Phi'(x)$ is bounded, the right hand side goes to 0 as $x \to +\infty$. Therefore, necessarily

$$A'(x)\Phi^2(x)\vartheta(x) = \frac{2}{\sigma^2}\int_x^{+\infty}\chi(\eta)d\eta.$$

Now, since $A(0) = 0$, we get

$$A(x) = \frac{2}{\sigma^2}\int_0^x \frac{1}{\Phi(\xi)}\left(\int_\xi^{+\infty}\frac{\chi(\eta)}{\chi(\xi)}d\eta\right)d\xi,$$

and therefore, $Z_0(x)$ is also given by formula

$$Z_0(x) = \frac{2}{\sigma^2}\int_0^x \frac{\Phi(x)}{\Phi(\xi)}\left(\int_\xi^{+\infty}\frac{\chi(\eta)}{\chi(\xi)}d\eta\right)d\xi, \quad x \geq 0,$$

which proves that the right sense is also bounded. Next the function defined by (16.3.24) is solution of (16.3.3).

Let us prove the uniqueness of a bounded solution. It is, of course, sufficient to prove that a bounded solution of

$$AH_s(x) + (\alpha+k)H_s(x) = 0, \quad x > s$$
$$H_s(s) = 0$$

is 0. Suppose that $H_s(x)$ becomes strictly positive. It cannot have a positive local maximum. Otherwise, if x^* is the smallest local maximum strictly greater than s then

$$H_s'(x^*) = 0, \quad H_s"(x^*) < 0,$$

and we get a contradiction. Therefore $H_s'(x) > 0$. Since $H_s(x)$ is bounded, we must have $H_s'(x) \to 0$, as $x \to +\infty$. Consider a point x_0 such that $\nu + k(x_0 - \gamma) > 0$. There exists a point x^* such that $H_s"(x^*) < 0$. At this point, there is again a contradiction. Therefore $H_s(x) \leq 0$. By a similar reasoning we see easily that $H_s(x)$ cannot become strictly negative. Hence it is 0. The proof has been completed. □

16.3.4. FUNCTION $G_s(x)$. Define first

$$Q(x) = \frac{2}{\sigma^2}\int_x^{+\infty} g'(\eta)\frac{\chi(\eta)}{\chi(x)}d\eta. \tag{16.3.27}$$

So the function $H_s(x)$ solution of (16.3.3) (s is a fixed parameter) can be written as follows

$$H_s(x) = \int_s^x \frac{\Phi(x)}{\Phi(\xi)}Q(\xi)d\xi. \tag{16.3.28}$$

From the function $H_s(x)$ we define the function $G_s(x)$ by the formula

$$G_s(x) = \begin{cases} G_s(s) + \int_s^x H_s(\xi)d\xi, & x > s \\ G_s(s), & x \leq s \end{cases}, \tag{16.3.29}$$

with the choice

$$G_s(s) = \frac{g(s) + \frac{\sigma^2}{2}Q(s) + c(\nu - k\gamma)}{\alpha}. \tag{16.3.30}$$

We then have the

Lemma 16.2. *The function $G_s(x)$ is C^1 and satisfies*

$$AG_s(x) + \alpha G_s(x) = g(x) + c(\nu - k\gamma),\ x > s$$

PROOF. The fact that it is C^1 is a consequence of $H_s(s) = 0$. Next the equation of $H_s(x)$ (16.3.3) can be written as follows

$$-\frac{\sigma^2}{2}H_s"(x) + \frac{d}{dx}[(\nu + k(x-\gamma))H_s(x)] + \alpha H_s(x) = g'(x).$$

Integrating between s and x and rearranging we get for $x > s$

$$-\frac{\sigma^2}{2}G_s"(x) + \frac{\sigma^2}{2}H_s'(s) + (\nu + k(x-\gamma))G_s'(x) + \alpha G_s(x) - \alpha G_s(s) = g(x) - g(s),$$

and from the definition of $G_s(s)$ the result follows, noting that $H_s'(s) = Q(s)$. □

Lemma 16.3. *The function $G_s(x)$ attains its infimum for $x \geq s$.*

PROOF. From the definition of $Q(x)$ (16.3.27) and the value of $g'(x)$ we get that $Q(x) > 0$ for $x \geq 0$. For $x < 0$ we have the formula

$$\frac{\sigma^2}{2}Q(x)\chi(x) = (-p + c(\alpha+k))\int_x^0 \chi(\eta)d\eta + (h + c(\alpha+k))\int_0^\infty \chi(\eta)d\eta,$$

which is clearly an increasing function. We recall, see Lemma 16.1, that $\chi(x) \to +\infty$ as $x \to -\infty$. Therefore

$$Q(x)\chi(x) \to -\infty, \text{ as } x \to -\infty$$

It follows that there exists a unique point $x_0 < 0$ such that

$$Q(x_0) = 0, \quad Q(x) > 0, \forall x > x_0, \quad Q(x) < 0, \forall x < x_0 < 0$$

For $s \geq x_0$, $H_s(x) > 0$, hence $G_s(x)$ increases. Therefore its minimum is at s.

Assume next $s < x_0 < 0$.

We consider $x > 0$. Then we can write

$$AH_s(x) + (\alpha + k)H_s(x) = h + c(\alpha + k),$$

therefore we can also write

$$H_s(x) = \frac{h + c(\alpha + k)}{\alpha + k} + \Phi(x)\left(H_s(0) - \frac{h + c(\alpha + k)}{\alpha + k}\right), \quad \forall x \geq 0.$$

This implies

$$H_s(x) \to \frac{h + c(\alpha + k)}{\alpha + k}, \text{ as } x \to +\infty$$

and thus

$$G_s(x) \to +\infty, \text{ as } x \to +\infty$$

Since $H_s'(s) = Q(s) < 0$, $H_s(x)$ remains strictly negative for x close to s, larger than s. Therefore $G_s(x)$ decreases for x close to s, larger than s.

Since it goes to $+\infty$ as $x \to +\infty$, it attains its minimum for $x \geq s$, at points $\neq s$. ☐

We can define $S(s)$ to be the smallest minimum of $G_s(x)$ for $x \geq s$. From the proof of Lemma 16.3. we see that

$$S(s) = s, \forall s \geq x_0, \quad S(s) > x_0, \forall s < x_0 \tag{16.3.31}$$

16.3.5. OBTAINING s. The point s is the solution of (16.3.4) which means also

$$0 = K + \int_s^{S(s)} H_s(\eta) d\eta. \tag{16.3.32}$$

We have the

Proposition 16.1. *There exists a unique $s < x_0$ solution of* (16.3.32).

Proof. We study the function

$$\gamma(s) = \int_s^{S(s)} H_s(\eta) d\eta.$$

Clearly

$$\gamma(s) = 0, \quad \forall s \geq x_0.$$

For $s < x_0$, we have, noting that $H_s(s) = H_s(S(s)) = 0$,

$$\gamma'(s) = \int_s^{S(s)} \frac{\partial}{\partial s} H_s(\eta) d\eta$$

and from formula (16.3.28) we get

$$\gamma'(s) = -\frac{Q(s)}{\Phi(s)} \int_s^{S(s)} \Phi(x) dx, \tag{16.3.33}$$

and since $Q(s) < 0$, we see immediately that $\gamma'(s) > 0$. The important step is to show that

$$\liminf_{s \to -\infty} \gamma'(s) \geq \frac{p - c(\alpha + k)}{\alpha + k}. \tag{16.3.34}$$

This will be done in the next Lemma. If (16.3.34) holds, then

$$\gamma(s) \to -\infty, \quad \text{as } s \to -\infty. \tag{16.3.35}$$

Since then, $\gamma(s)$ increases from $-\infty$ to 0, as s increases from $-\infty$ to 0. Therefore there exists a unique $s < x_0$ such that $\gamma(s) = -K$. This concludes the proof. ☐

The function $G_s(x)$ defined by (16.3.29) and (16.3.30) satisfies (16.3.1), (16.3.2). It remains to show the

Lemma 16.4. *The property* (16.3.34) *is satisfied.*

Proof. We first notice that

$$\gamma'(s) \geq -\frac{Q(s)}{\Phi(s)} \int_s^{x_0} \Phi(\eta) d\eta.$$

Also

$$\gamma'(s) \geq -\frac{Q(s)}{\Phi(s)} \int_s^0 \Phi(\eta) d\eta \left(1 - \frac{\int_{x_0}^0 \Phi(\eta) d\eta}{\int_s^0 \Phi(\eta) d\eta}\right). \tag{16.3.36}$$

Since

$$\int_s^0 \Phi(\eta) d\eta \to +\infty, \text{as } s \to -\infty$$

the second term in the parenthesis goes to 0 as $s \to -\infty$. So we focus on the function

$$Z(x) = -\frac{Q(x)}{\Phi(x)} \int_x^0 \Phi(\eta) d\eta,\ x \leq 0$$

We next note that

$$Q(x) = \frac{2}{\sigma^2}(-p + c(\alpha + k)) \int_x^0 \frac{\chi(\eta)}{\chi(x)} d\eta + \frac{2}{\sigma^2}(h + c(\alpha + k)) \int_0^{+\infty} \frac{\chi(\eta)}{\chi(x)} d\eta,$$

and since $\chi(x) \to +\infty$, as $x \to -\infty$, the second term goes to 0 as $x \to -\infty$. Consider next the equation (16.3.22) for $\chi(x)$ for $x < 0$. We integrate between x and 0 and divide by $\chi(x)$. We easily obtain

$$\int_x^0 \frac{\chi(\eta)}{\chi(x)} d\eta = -\frac{\sigma^2}{2(\alpha + k)} \frac{\Phi'(x)}{\Phi(x)} + \frac{\frac{\sigma^2}{2}\chi'(0) + \chi(0)(\nu - k\gamma)}{(\alpha + k)\chi(x)},$$

and the second term goes to 0 as $x \to -\infty$.

Similarly, considering the equation for $\Phi(x)$ for $x < 0$, see (17.2.1), we integrate between x and 0 and obtain

$$\int_x^0 \frac{\Phi(\eta)}{\Phi(x)} d\eta = -\frac{\sigma^2}{2\alpha} \frac{\Phi'(x)}{\Phi(x)} + \frac{\nu + k(x - \gamma)}{\alpha} + \frac{\frac{\sigma^2}{2}\Phi'(0) - \Phi(0)(\nu - k\gamma)}{\alpha\Phi(x)},$$

and the third term goes to 0 as $x \to -\infty$.

We are going to prove the two following properties

$$-\frac{\Phi'(x)}{\Phi(x)} \to +\infty, \text{as } x \to -\infty \tag{16.3.37}$$

$$\nu + k(x - \gamma) - \frac{\sigma^2}{2} \frac{\Phi'(x)}{\Phi(x)} \geq \frac{1}{2} B(x), \quad \forall x < \bar{x}, \tag{16.3.38}$$

for some $\bar{x} < 0$ and

$$B(x) = \frac{\nu + k(x - \gamma) + \sqrt{(\nu + k(x - \gamma))^2 + 2\alpha\sigma^2}}{2}.$$

Clearly

$$-B(x)(\nu + k(x - \gamma)) \to \frac{\alpha\sigma^2}{2}, \text{as } x \to -\infty. \tag{16.3.39}$$

Suppose we have obtained properties (16.3.37), (16.3.38). In considering the behavior of $Z(x)$ as $x \to -\infty$, we look for the dominating term. From the considerations above, we can write

$$Z(x) = \frac{1}{\alpha}\left[\frac{2}{\sigma^2}\frac{p-c(\alpha+k)}{\alpha+k}\left(-\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)} + \frac{\frac{\sigma^2}{2}\chi'(0)+\chi(0)(\nu-k\gamma)}{\chi(x)}\right)\right.$$
$$\left.-\frac{\frac{\sigma^2}{2}\chi'(0)+\chi(0)(\nu-k\gamma)}{\chi(x)}\right]\left[-\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)}+\nu+k(x-\gamma)\right.$$
$$\left.+\frac{\frac{\sigma^2}{2}\Phi'(0)-\Phi(0)(\nu-k\gamma)}{\Phi(x)}\right].$$

In the first bracket, from (16.3.37) it is clear that the dominating term is

$$\frac{p-c(\alpha+k)}{\alpha+k}\left(-\frac{\Phi'(x)}{\Phi(x)}\right).$$

In the second bracket the dominating term is not immediate. However

$$\frac{\frac{\sigma^2}{2}\Phi'(0)-(\nu-k\gamma)}{\Phi(x)} = \frac{\frac{\sigma^2}{2}\Phi'(0)-(\nu-k\gamma)}{\chi(x)}\vartheta(x).$$

Recall that

$$\vartheta(x) = \exp -\frac{2}{\sigma^2}\left[k\frac{(x-\gamma)^2}{2}+\nu x\right],$$

then

$$\frac{\frac{\sigma^2}{2}\Phi'(0)-(\nu-k\gamma)}{\Phi(x)} = B(x)\left[\frac{\frac{\sigma^2}{2}\Phi'(0)-(\nu-k\gamma)}{\chi(x)}\frac{-\vartheta(x)(\nu+k(x-\gamma))}{-B(x)(\nu+k(x-\gamma))}\right].$$

It follows from (16.3.39), the properties of $\chi(x)$ and the value of $\vartheta(x)$ that the function in the brackets tends to 0 as $x \to -\infty$. From (16.3.38) it follows easily that the dominant term is

$$-\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)}+\nu+k(x-\gamma),$$

therefore, collecting results we can assert that the dominant term in $Z(x)$ is

$$\tilde{Z}(x) = \frac{p-c(\alpha+k)}{\alpha(\alpha+k)}\left(-\frac{\Phi'(x)}{\Phi(x)}\right)\left(-\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)}+\nu+k(x-\gamma)\right).$$

Define

$$u(x) = -\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)}+\nu+k(x-\gamma),$$

then

$$\tilde{Z}(x) = \frac{2}{\sigma^2}\frac{p-c(\alpha+k)}{\alpha(\alpha+k)}u(u-(\nu+k(x-\gamma)\nu). \tag{16.3.40}$$

We now prove (16.3.37), (16.3.38), by studying u more precisely. First, from the relation

$$\frac{\sigma^2}{2}\Phi'(0) - (\nu - k\gamma) + \alpha \int_0^{+\infty} \Phi(x)dx = 0,$$

we deduce (recalling that $\Phi(0) = 1$)

$$u(0) > 0.$$

Moreover, one easily checks that $u(x)$ is the solution of the differential equation

$$u' = \frac{2u^2}{\sigma^2} - \frac{2u}{\sigma^2}(\nu + k(x-\gamma)) - \alpha, \quad u(0) > 0. \tag{16.3.41}$$

We first check that $u(x) > 0$ for $x < 0$. Indeed, we can write for $x < 0$

$$-\frac{\sigma^2}{2}\Phi'(0) + \frac{\sigma^2}{2}\Phi'(x) + \nu - k\gamma - (\nu + k(x-\gamma))\Phi(x) + \alpha \int_x^0 \Phi(\eta)d\eta = 0,$$

hence

$$\frac{-\frac{\sigma^2}{2}\Phi'(0) + (\nu - k\gamma)}{\Phi(x)} - u(x) + \alpha \int_x^0 \Phi(\eta)d\eta = 0,$$

which implies $u(x) > 0$, for $x < 0$. Next we can write (16.3.41) as

$$u' = \frac{2}{\sigma^2}(u(x) - B(x))(u(x) + C(x)), \tag{16.3.42}$$

with

$$C(x) = B(x) - (\nu + k(x-\gamma)).$$

Note that

$$B(x)C(x) = \frac{\alpha\sigma^2}{2}.$$

Consider now

$$v(x) = \frac{u(x)}{B(x)} > 0,$$

it satisfies the differential equation

$$v' = \frac{2}{\sigma^2}(v-1)\left(Bv + \frac{\alpha\sigma^2}{2B}\right) - v\frac{k}{\sqrt{(\nu + k(x-\gamma))^2 + 2\alpha\sigma^2}}, \quad x < 0. \tag{16.3.43}$$

We deduce

$$-\frac{1}{2}\frac{d}{dx}((1-v)^+)^2 = v'(1-v)^+ < 0,$$

therefore the function $((1-v)^+)^2$ increases on $(-\infty, 0)$. It follows that if there exists $\bar{x} < 0$ such that $(1-v)^+(\bar{x}) = 0$, then necessarily

$$(1-v)^+(x) = 0, \forall x < \bar{x}$$

Hence if $v(\bar{x}) \geq 1$, necessarily $v(x) \geq 1, \forall x < \bar{x}$. Suppose now that $v(x) < 1, \forall x < 0$, then from the equation (16.3.43) it follows that $v'(x) < 0$. Therefore $v(x)$ increases as $x \to -\infty$. It necessarily converges to $v^* \leq 1$. If $v^* < 1$, we see from the equation that $v'(x) \to -\infty$, as $x \to -\infty$. This is a contradiction. Summarizing we have the situation: either

$$v(x) \geq 1, \ \forall x < \bar{x}$$

or

$$v(x) \uparrow 1, \text{ as } x \downarrow -\infty$$

Since $u(x) = v(x)B(x)$ we have either

$$u(x) \geq B(x), \forall x < \bar{x}$$

or

$$u(x) - B(x) = (v(x) - 1)B(x) \to 0, \text{ as } x \to -\infty$$

So in both cases we can assert the property (16.3.38). Next, either

$$-\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)} \geq -(\nu + k(x - \gamma)) + B(x), \quad \forall x \leq \bar{x}$$

or

$$-\frac{\sigma^2}{2}\frac{\Phi'(x)}{\Phi(x)} = B(x) + u(x) - B(x) - (\nu + k(x - \gamma)) \to \infty, \text{as} \quad x \to -\infty.$$

In both cases we obtain property (16.3.37). So $\tilde{Z}(x)$ given by (16.3.40) is the dominant term in the expression of $Z(x)$. Now we can write

$$\begin{aligned}\tilde{Z}(x) &= \frac{2}{\sigma^2}\frac{p - c(\alpha + k)}{\alpha(\alpha + k)}Bv(Bv - (\nu + k(x - \gamma)\nu) \\ &= \frac{2}{\sigma^2}\frac{p - c(\alpha + k)}{\alpha(\alpha + k)}\left(B^2(x)v(x)(v(x) - 1) + v(x)\frac{\alpha\sigma^2}{2}\right).\end{aligned}$$

Now, since

$$\liminf_{x \to -\infty} v(x) \geq 1,$$

we can assert that

$$\liminf_{x \to -\infty} \tilde{Z}(x) = \frac{p - c(\alpha + k)}{\alpha + k},$$

hence also

$$\liminf_{x \to -\infty} Z(x) = \frac{p - c(\alpha + k)}{\alpha + k}$$

which proves (16.3.34), and concludes the proof. □

16.4. SOLUTION OF THE Q.V.I

It remains to show that the function $G_s(x)$ defined by (16.3.1), (16.3.2) is solution of the Q.V.I. (16.2.5). We can state the

Theorem 16.3. *The function $G_s(x)$ defined by* (16.3.1), (16.3.2) *is solution of the Q.V.I.* (16.2.5).

Proof. We first check the complementarity slackness condition (third condition (16.2.5)). For $x \leq s$, we have

$$\begin{aligned}G_s(x) &= G_s(s) \\ &= K + G_s(S(s)) \\ &= K + \inf_{\eta > s} G_s(\eta).\end{aligned}$$

But

$$\inf_{\eta > s} G_s(\eta) = \inf_{\eta > x} G_s(\eta) \quad \forall x \leq s.$$

Indeed, for $x < \eta < s$, one has

$$G_s(\eta) = G_s(s) > \inf_{\eta > s} G_s(\eta).$$

Therefore, we can assert that

$$G_s(x) = K + \inf_{\eta > x} G_s(\eta) \quad \forall x \leq s.$$

Clearly, the complementarity slackness condition is satisfied. It remains to show that

$$AG_s(x) + \alpha G_s(x) \leq g(x) + c(\nu - k\gamma), \quad x < s \tag{16.4.1}$$

$$G_s(x) \leq K + \inf_{\eta > x} G_s(\eta) \quad \forall x \geq s. \tag{16.4.2}$$

But (16.4.1) means

$$G_s(s) \leq g(x) + c(\nu - k\gamma), \quad x < s,$$

and from formula (16.3.30) we need to prove

$$g(s) + \frac{\sigma^2}{2} Q(s) \leq g(x), \quad x \leq s,$$

which is true since $Q(s) < 0$ and $x \leq s \leq 0$.

The proof of (16.4.2) is identical to that of (15.3.20), Theorem 15.2, Chapter 15. This concludes the proof. □

CHAPTER 17

TWO BAND IMPULSE CONTROL PROBLEMS

17.1. INTRODUCTION

In the inventory control problems considered in the previous chapters, the impulse control was a replenishment type ordering control. The inventory depleted by the demand had to be replenished from time to time (the impulse times). There are situations in which one may want to reduce the inventory. This will be the case when there is an additional source of supply (the mean reverting rule is one example). It may then not be optimal to accumulate inventory. A typical example is the cash management problem introduced by Constantinides and Richard [**17**]. For another application see Cadenillas and Zapatero [**15**]. This chapter is based on Bensoussan, Liu and Yuan [**10**].

17.2. THE PROBLEM

17.2.1. THE MODEL. We consider a probability space $\Omega, \mathcal{A}, P$ on which is defined a Wiener process $w(t)$. Let $\mathcal{F}^t = \sigma(w(s), s \leq t)$

An impulse control is a sequence

$$\theta_n, v_n,$$

where θ_n is a stopping time with respect to the filtration $\mathcal{F}^t$ and v_n is a random variable $\mathcal{F}^{\theta_n}$ measurable. Denoting by V an impulse control, the corresponding inventory is described by the formula

$$y_x(t; V) = x + \mu t + \sigma w(t) + M(t; V), \tag{17.2.1}$$

with

$$M(t) = M(t; V) = \sum_{\{n|\theta_n < t\}} v_n$$

In this model the drift term μt combines a mean demand and a fixed resupply policy. The number μ is any real number.

Let

$$f(x) = hx^+ + px^-.$$

We define the cost functional by

$$J_x(V) = E\left[\sum_{n=0}^{\infty} C(v_n) \exp -\alpha\theta_n + \int_0^{\infty} f(y_x(t; V)) \exp -\alpha t dt\right], \tag{17.2.2}$$

where

$$C(v) = K^+ \mathbb{1}_{v>0} + K^- \mathbb{1}_{v<0} + c^+ v^+ + c^- v^-, \tag{17.2.3}$$

in which $v^+ = \max(v, 0)$, $v^- = -\min(v, 0)$ and K^+, K^-, c^+, c^- are positive constants. Note that the upper symbols $+, -$ have different meanings.

Conversely to the previous models, the impulses can be positive or negative, and they are subject to different fixed and variable costs.

The value function is defined by

$$u(x) = \inf \ J_x(V).$$

17.2.2. Q.V.I. We write the Q.V.I arising from Dynamic Programming. We consider the operator

$$Au(x) = -\frac{1}{2}\sigma^2 u"(x) - \mu u'(x). \tag{17.2.4}$$

The Q.V.I. is given by

$$\begin{gathered} Au(x) + \alpha u(x) \leq f(x); \\ u(x) \leq M(u)(x); \\ (Au(x) + \alpha u(x) - f(x))(u(x) - M(u)(x)) = 0, \end{gathered} \tag{17.2.5}$$

with

$$M(u)(x) = \inf_{v \neq 0}[C(v) + u(x+v)].$$

We look for a function which is C^1 with linear growth. Standard arguments show that the solution of the Q.V.I is the value function.

17.2.3. TRANSFORMATION. Define

$$M^+u(x) = \inf_{v \geq 0}(K^+ 1\!\!1_{v>0} + c^+ v + u(x+v));$$

$$M^-u(x) = \inf_{v \leq 0}(K^- 1\!\!1_{v<0} - c^- v + u(x+v)),$$

then

$$Mu(x) = \min(M^+u(x), M^-u(x)),$$

and the Q.V.I. can be written as follows

$$\begin{gathered} Au(x) + \alpha u(x) \leq f(x); \\ u(x) \leq M^+u(x); \\ u(x) \leq M^-u(x); \\ (Au(x) + \alpha u(x) - f(x))(u(x) - M^+u(x))(u(x) - M^-u(x)) = 0. \end{gathered} \tag{17.2.6}$$

Introduce the functions

$$G^+(x) = u(x) + c^+ x; \tag{17.2.7}$$

$$G^-(x) = u(x) - c^- x; \tag{17.2.8}$$

$$g + (x) = f(x) + \alpha c^+ x; \tag{17.2.9}$$

$$g^-(x) = f(x) - \alpha c^- x, \tag{17.2.10}$$

then we can write the Q.V.I. in two parts

$$\begin{gathered} AG^+ + \alpha G^+ \leq g^+(x) - c^+\mu \\ G^+(x) \leq K^+ + \inf_{\eta > x} G^+(\eta) \\ (AG^+ + \alpha G^+ - g^+(x) + c^+\mu)(G^+(x) - K^+ - \inf_{\eta > x} G^+(\eta)) = 0 \\ \text{if} \quad G^-(x) < K^- + \inf_{\eta < x} G^-(\eta) \end{gathered} \tag{17.2.11}$$

$$
\begin{aligned}
&AG^- + \alpha G^- \leq g^-(x) + c^-\mu \\
&G^-(x) \leq K^- + \inf_{\eta<x} G^-(\eta) \\
&(AG^- + \alpha G^- - g^-(x) - c^-\mu)(G^-(x) - K^- - \inf_{\eta<x} G^-(\eta)) = 0 \\
&\text{if} \quad G^+(x) < K^+ + \inf_{\eta<x} G^+(\eta)
\end{aligned} \tag{17.2.12}
$$

$$G^+(x) - G^-(x) = (c^+ + c^-)x \tag{17.2.13}$$

17.3. a, A, b, B POLICY

17.3.1. FUNCTIONS $G^+_{ab}(x)$ AND $G^-_{ab}(x)$. Generalizing the idea of s, S policy, we introduce the concept of a, A, b, B policy. Let first $a < b$ be two real numbers we define the functions $G^+_{ab}(x)$ and $G^-_{ab}(x)$ by the following Q.V.I (note they are uncoupled)

$$
\begin{aligned}
&AG^+_{ab} + \alpha G^+_{ab} = g^+(x) - c^+\mu, && \forall a < x < b \\
&G^+_{ab}(x) = K^+ + \inf_{a<\eta<b} G^+_{ab}(\eta), && \forall x \leq a \\
&G^+_{ab}(b) = K^- + (c^+ + c^-)b + \inf_{a<\eta<b}(G^+_{ab}(\eta) - (c^+ + c^-)\eta)
\end{aligned} \tag{17.3.1}
$$

$$
\begin{aligned}
&AG^-_{ab} + \alpha G^-_{ab} = g^-(x) + c^-\mu, && \forall a < x < b \\
&G^-_{ab}(x) = K^- + \inf_{a<\eta<b} G^-_{ab}(\eta), && \forall x \geq b \\
&G^-_{ab}(a) = K^+ - (c^+ + c^-)a + \inf_{a<\eta<b}(G^-_{ab}(\eta) + (c^+ + c^-)\eta)
\end{aligned} \tag{17.3.2}
$$

The first problem is defined on the domain $(-\infty, b]$ and the second problem is defined on the domain $[a, +\infty)$. Considering the conditions on the common domain $[a, b]$ we obtain two-point boundary-value problems, with non-local Dirichlet conditions. On this domain, one has the relation

$$G^+_{ab}(x) - G^-_{ab}(x) = (c^+ + c^-)x. \tag{17.3.3}$$

From the second relation (17.3.1) we get an extension of $G^+_{ab}(x)$ below a and from the second equation (17.3.2) we get an extension of $G^-_{ab}(x)$ beyond b. Using (17.3.3) we define $G^+_{ab}(x)$ and $G^-_{ab}(x)$ on the whole line, and the relation (17.3.3) holds true. When $a < b$ are arbitrary numbers, we can only assert that $G^+_{ab}(x)$ and $G^-_{ab}(x)$ are continuous. To get C^1 functions on the whole line, we need to impose the conditions

$$(G^+_{ab})'(a) = 0, \ \ (G^-_{ab})'(b) = 0, \tag{17.3.4}$$

which can be considered as equations defining a and b.

17.3.2. FUNCTIONS $H^+_{ab}(x)$ AND $H^-_{ab}(x)$. The next step is to consider the functions

$$H^+_{ab}(x) = (G^+_{ab})'(x), \quad H^-_{ab}(x) = (G^-_{ab})'(x). \tag{17.3.5}$$

These functions are solutions of two-point boundary value problems

$$AH^+_{ab} + \alpha H^+_{ab} = f' + \alpha c^+, \ a < x < b, \quad H^+_{ab}(a) = 0, \ H^+_{ab}(b) = c^+ + c^- \tag{17.3.6}$$

$$AH^-_{ab} + \alpha H^-_{ab} = f' - \alpha c^-, \ a < x < b, \quad H^-_{ab}(a) = -(c^+ + c^-), \ H^-_{ab}(b) = 0, \tag{17.3.7}$$

and of course

$$H^+_{ab}(x) - H^-_{ab}(x) = c^+ + c^-. \tag{17.3.8}$$

We extend $H^+_{ab}(x)$ and $H^-_{ab}(x)$ beyond the interval (a, b) by constant values, so that (17.3.8) holds on the whole line. The functions $H^+_{ab}(x)$ and $H^-_{ab}(x)$ are continuous on the whole line, for any pair $a < b$. The system of equations to define the pair (a, b) becomes

$$F(a,b) = K^+ + \inf_{a \le x \le b} \int_a^x H_{ab}^+(\xi) d\xi = 0 \tag{17.3.9}$$

$$G(a,b) = -K^- + \sup_{a \le x \le b} \int_x^b H_{ab}^-(\xi) d\xi = 0. \tag{17.3.10}$$

We shall in the sequel make the assumptions

$$p - \alpha c^+ > 0, \quad h - \alpha c^- > 0. \tag{17.3.11}$$

It is fairly easy to give analytic formulas for $H_{ab}^+(x)$ and $H_{ab}^-(x)$. Let us consider the 2nd order equation

$$-\frac{1}{2}\sigma^2\beta^2 - \mu\beta + \alpha = 0,$$

and the roots

$$\beta_1 = -\frac{\mu - \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2} > 0, \quad \beta_2 = -\frac{\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2} < 0,$$

then we have the formulas

$$H_{ab}^+(x) = Z^+(a,b)\frac{\exp\beta_1(x-a) - \exp\beta_2(x-a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}$$
$$- \frac{2}{\sigma^2(\beta_1 - \beta_2)} \int_a^x (f'(\xi) + \alpha c^+)(\exp\beta_1(x-\xi) - \exp\beta_2(x-\xi))d\xi,$$

with

$$Z^+(a,b) = c^+ + c^-$$
$$+ \frac{2}{\sigma^2(\beta_1 - \beta_2)} \int_a^b (f'(\xi) + \alpha c^+)(\exp\beta_1(b-\xi) - \exp\beta_2(b-\xi))d\xi;$$

$$H_{ab}^-(x) = Z^-(a,b)\frac{\exp\beta_1(x-b) - \exp\beta_2(x-b)}{\exp\beta_1(a-b) - \exp\beta_2(a-b)}$$
$$+ \frac{2}{\sigma^2(\beta_1 - \beta_2)} \int_x^b (f'(\xi) - \alpha c^-)(\exp\beta_1(x-\xi) - \exp\beta_2(x-\xi))d\xi,$$

with

$$Z^-(a,b) = -(c^+ + c^-) + \frac{2}{\sigma^2(\beta_1 - \beta_2)} \int_a^b (f'(\xi) - \alpha c^-)(\exp\beta_1(a-\xi) - \exp\beta_2(a-\xi))d\xi.$$

17.3.3. FUNCTIONS $A(a, b)$ AND $B(a, b)$.

We begin with

Lemma 17.1. *We assume $a < 0 < b$. Under the first assumption* (17.3.11) *the function $Z^+(a, b)$ is increasing in a. Moreover $Z^+(0, b) > 0, Z^+(-\infty, b) = -\infty$. Similarly, under the second assumption* (17.3.11)*, the function $Z^-(a, b)$ is increasing in b. Moreover $Z^-(a, 0) < 0$ and $Z^-(a, +\infty) = +\infty$.*

PROOF. We have

$$Z^+(a,b) = -\frac{h - \alpha c^-}{\alpha} + \frac{1}{\alpha(\beta_1 - \beta_2)}[(p+h)(\beta_1 \exp\beta_2 b - \beta_2 \exp\beta_1 b)$$
$$-(p - \alpha c^+)(\beta_1 \exp\beta_2(b-a) - \beta_2 \exp\beta_1(b-a))],$$

hence

$$\frac{\partial Z^+}{\partial a} = \frac{2}{\sigma^2(\beta_1-\beta_2)}(p-\alpha c^+)(\exp\beta_1(b-a)-\exp\beta_2(b-a)) > 0.$$

Clearly $Z^+(0,b) > 0$. Also from which we deduce $Z^+(-\infty,b) = -\infty$. Similarly

$$Z^-(a,b) = \frac{p-\alpha c^+}{\alpha} + \frac{1}{\alpha(\beta_1-\beta_2)}[-(p+h)(\beta_1\exp\beta_2 a - \beta_2\exp\beta_1 a) -$$

$$+(h-\alpha c^-)(\beta_1\exp\beta_2(a-b)-\beta_2\exp\beta_1(a-b))], \tag{17.3.12}$$

hence

$$\frac{\partial Z^-}{\partial b} = -\frac{2}{\sigma^2(\beta_1-\beta_2)}(h-\alpha c^-)(\exp\beta_1(a-b)-\exp\beta_2(a-b)) > 0,$$

and $Z^-(a,0) < 0$, $Z^-(a,+\infty) = +\infty$. □

Proposition 17.1. *We assume* (17.3.11) *and $a < 0 < b$. There exist unique numbers $A(a,b), B(a,b)$ such that*

$$a \le A(a,b) < B(a,b) \le b,$$

such that

$$H^+_{ab}(x) > 0, \forall x > A(a,b);\ H^+_{ab}(x) < 0, \forall a < x < A(a,b);\ H^+_{ab}(A) = 0$$

$$H^-_{ab}(x) > 0, \forall b > x > B(a,b);\ H^-_{ab}(x) < 0, \forall x < B(a,b);\ H^-_{ab}(B) = 0.$$

The cases $A(a,b) = a$ or $B(a,b) = b$ are possible, in which case $H^+_{ab}(x) > 0, \forall x > a$ or $H^-_{ab}(x) < 0, \forall x < b$.

Proof. From Lemma 17.1 it follows that there exist uniquely defined numbers $a_0(b) < 0$ and $b_0(a) > 0$ such that

$$Z^+(a,b) < 0, \forall a < a_0(b);\ Z^+(a,b) > 0, \forall a_0(b) < a < 0 \tag{17.3.13}$$

$$Z^-(a,b) > 0, \forall b > b_0(a);\ Z^-(a,b) < 0, \forall 0 < b < b_0(a). \tag{17.3.14}$$

Next we obtain from the formula of $H^+_{ab}(x)$

$$(H^+_{ab})'(a) = \frac{Z^+(a,b)(\beta_1-\beta_2)}{\exp\beta_1(b-a)-\exp\beta_2(b-a)},$$

therefore $(H^+_{ab})'(a)$ and $Z^+(a,b)$ have the same sign. We deduce that

$$a_0(b) \le a \Longrightarrow (H^+_{ab})'(a) \ge 0,$$

we then state that $H^+_{ab}(x) > 0, \forall x \in (a,b]$. Indeed there exists a small interval $(a, a+\epsilon)$ in which $H^+_{ab}(x) > 0$. This is clear when $a > a_0(b)$ since $(H^+_{ab})'(a) > 0$. It is also true when $a = a_0(b)$, since $(H^+_{ab})'(a) = 0$ and

$$-\frac{1}{2}\sigma^2(H^+_{ab})''(a) = -(p-\alpha c^+) < 0.$$

There cannot exist a point $\bar{x}$ with $a < \bar{x} < b$ such that $H^+_{ab}(\bar{x}) = 0$. Indeed, if such a point exists, we can consider the smallest such point, hence one has

$$H^+_{ab}(x) > 0, \forall x \in (a,\bar{x}).$$

Necessarily $\bar{x} > 0$. Otherwise since there is a positive maximum in $(a,\bar{x})$ there will be a local positive maximum in $(a,0)$ which is impossible, by maximum principle considerations. Moreover $(H^+_{ab})'(\bar{x}) < 0$. Since $H^+_{ab}(b) > 0$, the function $H^+_{ab}(x)$

has a negative minimum in $(0, b)$, which is also impossible by maximum principle considerations. So we have proven that $H^+_{ab}(x) > 0, \forall x \in (a, b]$. Defining

$$A(a, b) = a, \forall a \geq a_0(b),$$

the result is obtained for $H^+_{ab}(x)$. We can thus assume $a < a_0(b)$. We have then $(H^+_{ab})'(a) < 0$. This implies that for a small interval $(a, a + \epsilon)$, $H^+_{ab}(x) < 0$. Since $H^+_{ab}(b) > 0$, the function $H^+_{ab}(x)$ must vanish in (a, b). We can define $A(a, b)$ as the smallest point in (a, b) such that $H^+_{ab}(A) = 0$. In fact, we are going to prove that this point is unique and $H^+_{ab}(x) > 0, \forall x \in (A, b]$. We have $(H^+_{ab})'(A) > 0$. Therefore for a small interval $(A, A+\epsilon]$ we have $H^+_{ab}(x) > 0$. Consider the minimum of $H^+_{ab}(x)$ on $[A+\epsilon, b]$. Suppose this minimum is ≤ 0, and let x^* be this minimum. Then $x^* \leq 0$, by maximum principle considerations. But then $A < 0$ and there exists a local maximum of $H^+_{ab}(x)$ in the open interval $(A, 0)$. This is impossible, by maximum principle considerations. Therefore, for $a < a_0(b)$ there exists a unique $A(a, b)$ with $a < A(a, b) < b$ and

$$H^+_{ab}(x) > 0, \forall x > A(a, b);\ H^+_{ab}(x) < 0, \forall a < x < A(a, b);\ H^+_{ab}(A) = 0.$$

Similarly, we can check

$$b \leq b_0(a) \Longrightarrow H^-_{ab}(x) < 0, \forall x \in [a, b).$$

In this case we set $B(a, b) = b$. Otherwise if $b > b_0(a)$ there exists a unique $B(a, b)$ with $a < B(a, b) < b$ and

$$H^-_{ab}(x) > 0, \forall b > x > B(a, b);\ H^-_{ab}(x) < 0, \forall x < B(a, b);\ H^-_{ab}(B) = 0.$$

In view of (17.3.8) we have clearly

$$A(a, b) < B(a, b).$$

The proof has been completed. □

17.3.4. FUNCTIONS $a(b)$ AND $b(a)$. We recall the definition of functions $F(a, b)$,$G(a, b)$, see (17.3.9), (17.3.10). We are going to prove the

Proposition 17.2. *We assume* (17.3.11) *and* $a < 0 < b$. *There exists a unique* $a(b) < a_0(b)$ *such that*

$$F(a(b), b) = 0, \tag{17.3.15}$$

and a unique $b(a) > b_0(a)$ *such that*

$$G(a, b(a)) = 0. \tag{17.3.16}$$

To prove the Proposition, we shall rely on intermediary properties, interesting in themselves

Lemma 17.2. *For* $a \leq x \leq b$, *we have the properties*

$$\frac{\partial H^+_{ab}(x)}{\partial a} > 0, \forall a < a_0(b);\ \frac{\partial H^+_{ab}(x)}{\partial a} < 0, \forall a > a_0(b);\ \frac{\partial H^+_{ab}(x)}{\partial a}|_{a=a_0(b)} = 0 \tag{17.3.17}$$

$$\frac{\partial H^-_{ab}(x)}{\partial b} > 0, \forall b > b_0(a);\ \frac{\partial H^-_{ab}(x)}{\partial b} < 0, \forall b < b_0(a);\ \frac{\partial H^-_{ab}(x)}{\partial b}|_{b=b_0(a)} = 0. \tag{17.3.18}$$

PROOF. From the formula of $H^+_{ab}(x)$ we get after easy, although tedious calculations

$$\text{(17.3.19)} \quad \frac{\partial H^+_{ab}(x)}{\partial a} = Z^+(a,b)\frac{\exp\beta_1(x-b)-\exp\beta_2(x-b)}{(\exp\beta_1(a-b)-\exp\beta_2(a-b))^2}(\beta_1-\beta_2) \cdot \exp(\beta_1+\beta_2)(a-b),$$

and similarly

$$\text{(17.3.20)} \quad \frac{\partial H^-_{ab}(x)}{\partial b} = Z^-(a,b)\frac{\exp\beta_1(x-a)-\exp\beta_2(x-a)}{(\exp\beta_1(b-a)-\exp\beta_2(b-a))^2}(\beta_1-\beta_2) \cdot \exp(\beta_1+\beta_2)(b-a).$$

Therefore, from (17.3.13) and (17.3.14) we can assert that
(17.3.21)

$$\frac{\partial H^+_{ab}(x)}{\partial a} > 0, \text{ if } a < a_0(b); \quad \frac{\partial H^+_{ab}(x)}{\partial a} < 0, \text{ if } a > a_0(b); \quad \frac{\partial H^+_{ab}(x)}{\partial a}|_{a=a_0(b)}$$

$$\text{(17.3.22)} \quad \frac{\partial H^-_{ab}(x)}{\partial b} > 0, \text{ if } b > b_0(a); \quad \frac{\partial H^-_{ab}(x)}{\partial b} < 0, \text{ if } b < b_0(a); \quad \frac{\partial H^-_{ab}(x)}{\partial b}|_{b=b_0(a)}$$

PROOF OF PROPOSITION 17.2:

Consider next $F(a,b)$ and $G(a,b)$ defined by (17.3.9) and (17.3.10). We first note that

$$\text{(17.3.23)} \quad F(a,b) = K^+, \text{if } a \geq a_0(b); \quad G(a,b) = -K^-, \text{if } b \leq b_0(a).$$

Next, for $a < a_0(b)$

$$\frac{\partial F(a,b)}{\partial a} = \int_a^{A(a,b)} \frac{\partial H^+_{ab}(x)}{\partial a}dx > 0,$$

from (17.3.21). Similarly, for $b > b_0(a)$ one has

$$\frac{\partial G(a,b)}{\partial b} = \int_{B(a,b)}^{b} \frac{\partial H^-_{ab}(x)}{\partial b}dx > 0.$$

Hence $F(a,b)$ is strictly increasing in a on $(-\infty, a_0(b))$ and constant on $[a_0(b), 0]$. Similarly $G(a,b)$ is constant on $[0, b_0(a)]$ and strictly increasing in b on $(b_0(a), +\infty)$. We proceed by showing that

$$\text{(17.3.24)} \quad F(-\infty, b) = -\infty, \quad G(a, +\infty) = +\infty.$$

The monotonicity properties and (17.3.23), (17.3.24) imply immediately the existence and uniqueness of $a(b)$ and $b(a)$ such that (17.3.15) and (17.3.16) hold. There remains to prove (17.3.24). We shall prove only the first part. The second part is similar. It is useful to give explicit formulas for $H^+_{ab}(x)$ and $H^-_{ab}(x)$. One gets for $x > 0$,

$$H^+_{ab}(x) = (c^+ + c^-)\frac{\exp\beta_1(x-a) - \exp\beta_2(x-a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}$$
$$-\frac{h+\alpha c^+}{\alpha}\ \frac{\exp\beta_1(x-a) - \exp\beta_2(x-a) - \exp\beta_1(b-a) + \exp\beta_2(b-a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}$$
$$-\frac{p+h}{\alpha(\beta_1-\beta_2)}\ \frac{(\exp(\beta_2 x + \beta_1 b) - \exp(\beta_1 x + \beta_2 b))(\beta_1 \exp -\beta_1 a - \beta_2 \exp -\beta_2 a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}$$
$$+\frac{p-\alpha c^+}{\alpha}\exp-(\beta_1+\beta_2)a\ \frac{\exp(\beta_2 x + \beta_1 b) - \exp(\beta_1 x + \beta_2 b)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}$$

and for $x < 0$,

$$H^+_{ab}(x) = \frac{\exp\beta_1(x-a) - \exp\beta_2(x-a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}\Big[-\frac{h-\alpha c^-}{\alpha}$$
$$+\frac{p+h}{\alpha}\ \frac{\beta_1\exp\beta_2 b - \beta_2\exp\beta_1 b}{\beta_1-\beta_2}\Big] + \frac{p-\alpha c^+}{\alpha}$$
$$\cdot\frac{\exp\beta_1(b-a)(\exp\beta_2(x-a)-1) - \exp\beta_2(b-a)(\exp\beta_1(x-a)-1)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}.$$

Similarly we state:

For $x > 0$,

$$H^-_{ab}(x) = \frac{\exp\beta_1(x-b) - \exp\beta_2(x-b)}{\exp\beta_1(a-b) - \exp\beta_2(a-b)}\Big[\frac{p-\alpha c^+}{\alpha}$$
$$\cdot - \frac{p+h}{\alpha}\ \frac{\beta_1\exp\beta_2 a - \beta_2\exp\beta_1 a}{\beta_1-\beta_2}\Big] + \frac{h-\alpha c^-}{\alpha}$$
$$\cdot\frac{\exp\beta_2(a-b)(\exp\beta_1(x-b)-1) - \exp\beta_1(a-b)(\exp\beta_2(x-b)-1)}{\exp\beta_1(a-b) - \exp\beta_2(a-b)},$$

and for $x < 0$,

$$H^-_{ab}(x) = -(c^+ + c^-)\frac{\exp\beta_1(x-b) - \exp\beta_2(x-b)}{\exp\beta_1(a-b) - \exp\beta_2(a-b)} + \frac{p+\alpha c^-}{\alpha}$$
$$\cdot\frac{\exp\beta_1(x-b) - \exp\beta_2(x-b) - \exp\beta_1(a-b) + \exp\beta_2(a-b)}{\exp\beta_1(a-b) - \exp\beta_2(a-b)} + \frac{p+h}{\alpha(\beta_1-\beta_2)}$$
$$\cdot\frac{(\exp(\beta_2 x + \beta_1 a) - \exp(\beta_1 x + \beta_2 a))(\beta_1\exp-\beta_1 b - \beta_2\exp-\beta_2 b)}{\exp\beta_1(a-b) - \exp\beta_2(a-b)}$$
$$-\frac{h-\alpha c^-}{\alpha}\exp-(\beta_1+\beta_2)b\ \frac{\exp(\beta_2 x + \beta_1 a) - \exp(\beta_1 x + \beta_2 a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)}.$$

We thus deduce immediately:

For $x \leq 0$,

$$H^+_{-\infty b}(x) = -\frac{h-\alpha c^-}{\alpha}\exp\beta_1(x-b) - \frac{p-\alpha c^+}{\alpha}$$
$$+\frac{h+p}{\alpha(\beta_1-\beta_2)}\exp\beta_1(x-b)(\beta_1\exp\beta_2 b - \beta_2\exp\beta_1 b),$$

hence

$$H^+_{-\infty b}(x) \to -\frac{p-\alpha c^+}{\alpha} \quad \text{as } x \to -\infty,$$

and $H^+_{-\infty b}(b) = c^+ + c^-$. Therefore the point $A(-\infty, b)$ is well defined and finite. It follows that

$$\int_{-\infty}^{A(-\infty,b)} H^+_{-\infty b}(\xi)d\xi = -\infty,$$

hence the first part of (17.3.24) is obtained. The second part is proven in a similar manner. The proof of the proposition has been completed. □

17.3.5. OBTAINING a AND b. We look for points $a \leq 0$ and $b \geq 0$ such that (17.3.9), (17.3.10) hold. We begin by giving full expressions for $F(a,b)$, $G(a,b)$.

We know, of course, (17.3.23). So we consider $a < a_0(b)$ and $b > b_0(a)$. We have

$$F(a,b) = K^+ + \int_a^A H^+_{ab}(x)dx.$$

From equation (17.3.6) we obtain

$$-\frac{\sigma^2}{2}((H^+_{ab})'(A) - (H^+_{ab})'(a)) + \alpha \int_a^A H^+_{ab}(x)dx = \int_a^A (f'(x) + \alpha c^+)dx.$$

Using $\beta_1\beta_2 = -\dfrac{2\alpha}{\sigma^2}$ we have also

$$\int_a^A H^+_{ab}(x)dx = \frac{1}{\alpha}\int_a^A (f'(x) + \alpha c^+)dx - \frac{1}{\beta_1\beta_2}((H^+_{ab})'(A) - (H^+_{ab})'(a)).$$

Next, from the formula of $H^+_{ab}(x)$ we can assert

(17.3.25)
$$\begin{aligned}(H^+_{ab})'(x) = {} & Z^+(a,b)\frac{\beta_1 \exp\beta_1(x-a) - \beta_2 \exp\beta_2(x-a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)} \\ & -\frac{2}{\sigma^2(\beta_1-\beta_2)}\int_a^x (f'(\xi) + \alpha c^+)(\beta_1 \exp\beta_1(x-\xi) \\ & -\beta_2 \exp\beta_2(x-\xi))d\xi\end{aligned}$$

hence

$$\begin{aligned}(H^+_{ab})'(A) - (H^+_{ab})'(a) = {} & Z^+(a,b)\frac{(\beta_1(\exp\beta_1(A-a)-1) - \beta_2(\exp\beta_2(A-a)-1))}{\exp\beta_1(b-a) - \exp\beta_2(b-a)} \\ & -\frac{2}{\sigma^2(\beta_1-\beta_2)}\int_a^A (f'(\xi)+\alpha c^+)(\beta_1\exp\beta_1(A-\xi) - \beta_2\exp\beta_2(A-\xi))d\xi.\end{aligned}$$

On the other hand, since $H^+_{ab}(A) = 0$, we can also write

$$\begin{aligned}& Z^+(a,b)\frac{\exp\beta_1(A-a) - \exp\beta_2(A-a)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)} \\ & = \frac{2}{\sigma^2(\beta_1-\beta_2)}\int_a^A (f'(\xi)+\alpha c^+)(\exp\beta_1(A-\xi) - \exp\beta_2(A-\xi))d\xi.\end{aligned}$$

Combining, we get

$$\begin{aligned}(H^+_{ab})'(A) - (H^+_{ab})'(a) = {} & \frac{2}{\sigma^2}\int_a^A (f'(x)+\alpha c^+) \\ & \cdot\frac{\exp\beta_1(A-x)(\exp\beta_2(A-a)-1) - \exp\beta_2(A-x)(\exp\beta_1(A-a)-1)}{\exp\beta_1(A-a) - \exp\beta_2(A-a)}dx.\end{aligned}$$

Therefore

$$F(a,b) = K^+ + \frac{1}{\alpha}\int_a^A (f'(x)+\alpha c^+)dx + \frac{1}{\alpha}\int_a^A (f'(x)+\alpha c^+)$$
$$\cdot \frac{\exp\beta_1(A-x)(\exp\beta_2(A-a)-1)-\exp\beta_2(A-x)(\exp\beta_1(A-a)-1)}{\exp\beta_1(A-a)-\exp\beta_2(A-a)}dx.$$

Computing the integrals, we finally get

$$F(a,b) = K^+ + \frac{1}{\alpha}((-p+\alpha c^+)(A-a)+(h+p)A^+)$$
$$+\frac{1}{\alpha}\frac{p-\alpha c^+}{\beta_1\beta_2}(\beta_1-\beta_2)\frac{(\exp\beta_1(A-a)-1)(\exp\beta_2(A-a)-1)}{\exp\beta_1(A-a)-\exp\beta_2(A-a)} + \frac{1}{\alpha}\frac{p+h}{\beta_1\beta_2}$$
$$\cdot\frac{\beta_1(\exp\beta_1(A-a)-1)(1-\exp\beta_2A^+)-\beta_2(\exp\beta_2(A-a)-1)(1-\exp\beta_1A^+)}{\exp\beta_1(A-a)-\exp\beta_2(A-a)}.$$

We can, in a similar way give the formula for $G(a,b)$

$$G(a,b) = -K^- + \frac{1}{\alpha}((h-\alpha c^-)(b-B)-(h+p)B^-)$$
$$+\frac{1}{\alpha}\frac{h-\alpha c^-}{\beta_1\beta_2}(\beta_1-\beta_2)\frac{(\exp\beta_1(B-b)-1)(\exp\beta_2(B-b)-1)}{\exp\beta_1(B-b)-\exp\beta_2(B-b)} + \frac{1}{\alpha}\frac{p+h}{\beta_1\beta_2}$$
$$\cdot\frac{\beta_1(\exp\beta_1(B-b)-1)(1-\exp-\beta_2B^-)-\beta_2(\exp\beta_2(B-b)-1)(1-\exp-\beta_1B^-)}{\exp\beta_1(B-b)-\exp\beta_2(B-b)}$$

We can state the

Proposition 17.3. *We assume* (17.3.11)*. There exists a solution $a<0<b$ of the system* (17.3.9)*,* (17.3.10)*.*

Proof. The functions $A(a,b), B(a,b)$ and $F(a,b), G(a,b)$ are continuous. Moreover we have (17.3.24). It follows that the functions $a(b)$ and $b(a)$ are continuous. Indeed, if $b_n \to b$ and $a_n = a(b_n)$ then the sequence a_n must be bounded. Otherwise there would exist a subsequence converging to $-\infty$, but then, for this subsequence

$$F(a_n,b_n) \to F(-\infty,b) = -\infty,$$

which is impossible, since $F(a_n,b_n)=0$. It follows easily that the unique accumulation point of the sequence a_n, b_n is $a(b), b$. Therefore $a(b_n) \to a(b)$, which proves the continuity of the map $b \to a(b)$. A similar proof holds for the map $a \to b(a)$. Now a solution of (17.3.9), (17.3.10) satisfies

$$u(a) = a(b(a)) - a = 0.$$

The function $u(a)$ is continuous and $u(0) = a(b(0)) < 0$. Let us check that

$$u(-\infty) = +\infty. \tag{17.3.26}$$

If this is true, then the function $u(a)$ crosses the line 0 and there exists $\hat{a}$ such that $u(\hat{a})=0$. Then $\hat{a}$ and $\hat{b} = b(\hat{a})$ is a solution of (17.3.9), (17.3.10). To prove (17.3.26) it is sufficient to check that $b(-\infty)$ is finite. However if $b(-\infty)=+\infty$ then $G(-\infty,+\infty)=0$. This is not possible from (17.3.24). The proof has been completed. □

Remark 17.1. We do not claim the uniqueness of the point a,b although it is very likely.

17.4. SOLUTION OF THE Q.V.I.

From the a, A, B, b policy, we are going to construct $G^+(x), G^-(x)$ satisfying (17.2.11), (17.2.12), (17.2.13). We define

$$G^+_{ab}(x) = G^+_{ab}(a), \forall x \leq a; \;\; G^+_{ab}(x) = G^+_{ab}(a) + \int_a^x H^+_{ab}(\xi)d\xi, \forall x \geq a \tag{17.4.1}$$

$$G^-_{ab}(x) = G^-_{ab}(b), \forall x \geq b; \;\; G^-_{ab}(x) = G^-_{ab}(b) - \int_x^b H^-_{ab}(\xi)d\xi, \forall x \leq b, \tag{17.4.2}$$

with

$$G^+_{ab}(a) = \frac{g^+(a) - c^+\mu + \frac{\sigma^2}{2}(H^+_{ab})'(a)}{\alpha}; \tag{17.4.3}$$

$$G^-_{ab}(b) = \frac{g^-(b) + c^-\mu + \frac{\sigma^2}{2}(H^-_{ab})'(b)}{\alpha}. \tag{17.4.4}$$

We first check that

$$G^+_{ab}(x) - G^-_{ab}(x) = (c^+ + c^-)x. \tag{17.4.5}$$

Since

$$(G^+_{ab})'(x) - (G^-_{ab})'(x) = H^+_{ab}(x) - H^-_{ab}(x) = c^+ + c^-,$$

it is sufficient to check that

$$G^+_{ab}(a) - G^-_{ab}(a) = (c^+ + c^-)a. \tag{17.4.6}$$

Using

$$\begin{aligned} G^-_{ab}(a) &= G^-_{ab}(b) - \int_a^b H^-_{ab}(\xi)d\xi \\ &= G^-_{ab}(b) - \int_a^b H^+_{ab}(\xi)d\xi + (c^+ + c^-)(b-a), \end{aligned}$$

and formulas (17.4.3), (17.4.4) we must check the relation

$$\begin{aligned} & g^+(a) - c^+\mu + \frac{\sigma^2}{2}(H^+_{ab})'(a) \\ & \quad = g^-(b) + c^-\mu + \frac{\sigma^2}{2}(H^-_{ab})'(b) - \alpha \int_a^b H^+_{ab}(\xi)d\xi + \alpha(c^+ + c^-)b, \end{aligned}$$

or

$$-\frac{\sigma^2}{2}((H^-_{ab})'(b) - (H^+_{ab})'(a)) + \alpha \int_a^b H^+_{ab}(\xi)d\xi - (c^+ + c^-)\mu = f(b) - f(a) + \alpha c^+(b-a),$$

which follows immediately from (17.3.6) after integrating between a and b. The functions $G^+_{ab}(x)$ and $G^-_{ab}(x)$ satisfy (17.3.1), (17.3.2), (17.3.3). We then define

$$G^+(x) = G^+_{ab}(x), \quad G^-(x) = G^-_{ab}(x).$$

We note that

$$\begin{aligned} AG^+(x) + \alpha G^+(x) &= g^+(x) - c^+\mu, && \forall a < x < b \\ G^+(x) &= G^+_{ab}(a), && \forall x \leq a \\ G^+(x) &= G^-_{ab}(b) + (c^+ + c^-)x, && \forall x \geq b \end{aligned}$$

Therefore for $x < a$

$$AG^+(x) + \alpha G^+(x) = g^+(a) - c^+\mu + \frac{\sigma^2}{2}(H^+_{ab})'(a).$$

We use next $(H^+_{ab})'(a) < 0$ and

$$\begin{aligned} g^+(x) &= f(x) + \alpha c^+ x = (-p + \alpha c^+)x \\ &\geq (-p + \alpha c^+)a = g^+(a) \end{aligned}$$

to conclude that

$$AG^+(x) + \alpha G^+(x) \leq g^+(x) - c^+\mu, \quad \forall\, x < a.$$

For $x > b$, we have

$$\begin{aligned} AG^+(x) + \alpha G^+(x) &= -\mu(c^+ + c^-) + \alpha(c^+ + c^-)x \\ &+ g^-(b) + c^-\mu + \frac{\sigma^2}{2}(H^-_{ab})'(b). \end{aligned}$$

We then use $(H^-_{ab})'(b) < 0$ and

$$\begin{aligned} g^-(b) &= f(b) - \alpha c^- b = (h - \alpha c^-)b \leq (h - \alpha c^-)b \\ &= g^-(x) = g^+(x) - \alpha(c^+ + c^-)x, \end{aligned}$$

to conclude also that

$$AG^+(x) + \alpha G^+(x) \leq g^+(x) - c^+\mu, \quad \forall\, x > b.$$

Next we note that, from the choice of a, b we have

$$G^+_{ab}(a) = K^+ + \inf_{a<\eta<b} G^+_{ab}(\eta)$$

For $x \leq a$,

$$\begin{aligned} G^+(x) = G^+_{ab}(a) &= K^+ + \inf_{a<\eta<b} G^+(\eta) \\ &= K^+ + \inf_{x<\eta<b} G^+(\eta) = K^+ + \inf_{x<\eta} G^+(\eta). \end{aligned}$$

For $x > a$, to prove the same inequality we have to check that

$$0 \leq K^+ + \inf_{x<\eta} \int_x^\eta H^+_{ab}(\xi)d\xi. \tag{17.4.7}$$

This is obvious for $x > A$, since the integral is positive. So we may consider the situation $a \leq x \leq A$. But then

$$\begin{aligned} K^+ + \inf_{x<\eta} \int_x^\eta H^+_{ab}(\xi)d\xi &= K^+ + \int_x^A H^+_{ab}(\xi)d\xi \\ &\geq K^+ + \int_a^A H^+_{ab}(\xi)d\xi = 0. \end{aligned}$$

Therefore (17.4.7) is obtained. Considering (17.2.11) we see that the two first inequalities are satisfied. In a similar way, we check that the two first inequalities of (17.2.12) are also satisfied. Finally, considering the three intervals, $(-\infty, a), (a, b)$, $(b + \infty)$ we see that the third relations in (17.4.7) and in (17.2.12) are also satisfied. Therefore we have found a solution of the Q.V.I. (17.2.11), (17.2.12), (17.2.13) and thus also of (17.2.6) through a a, A, B, b policy.

17.5. COMPUTATIONAL ASPECTS

In this section we focus on computational aspects related to obtaining the four numbers a, A, B, b.

17.5.1. SIGN OF A AND B. We know that $A < B$ and $a < 0,\ b > 0$ but we cannot guarantee the sign of A, B. It is interesting to further study this question. It also leads to simplifications in the computation of the four numbers. Consider finding $a(b)$ for fixed b. We have to solve $F(a(b), b) = 0$. But looking at formula (17.5.13) we see that $F(a.b)$ depends on b only through $A(a, b)$. So for b fixed the pair $a(b)$,$A(a(b), b)$ is obtained via a system

$$F(a,b) = 0,\ H^+_{ab}(A) = 0. \tag{17.5.1}$$

In solving this system, we have different calculations depending on the sign of the solution A that we look for. If we postulate $A < 0$, then the pair a, A becomes the solution of the following system (recall that we omit to write b which is a fixed positive parameter in this framework)

$$K^+ - \frac{1}{\alpha}(p - \alpha c^+)(A - a) + \frac{1}{\alpha}\frac{p - \alpha c^+}{\beta_1\beta_2}(\beta_1 - \beta_2) \cdot \frac{(\exp\beta_1(A-a) - 1)(\exp\beta_2(A-a) - 1)}{\exp\beta_1(A-a) - \exp\beta_2(A-a)} = 0$$

$$h - \alpha c^- - (p+h)\frac{\beta_1\exp\beta_2 b - \beta_2\exp\beta_1 b}{\beta_1 - \beta_2} = (p - \alpha c^+) \cdot \frac{\exp\beta_1(b-a)(\exp\beta_2(A-a) - 1) - \exp\beta_2(b-a)(\exp\beta_1(A-a) - 1)}{\exp\beta_1(A-a) - \exp\beta_2(A-a)}.$$

Moreover this system is sequential. The first equation defines $A - a$ and is independent of b. The second equation defines a as a function of b and the value of $A - a$ obtained from the first one. If we define the function

$$Z(x) = -x + \frac{\beta_1 - \beta_2}{\beta_1\beta_2}\frac{(\exp\beta_1 x - 1)(\exp\beta_2 x - 1)}{\exp\beta_1 x - \exp\beta_2 x}, \tag{17.5.2}$$

then the first condition is equivalent to

$$Z(A - a) = \frac{-K^+\alpha}{p - \alpha c^+}. \tag{17.5.3}$$

We have the

Lemma 17.3. *There exists a unique number $\varpi > 0$ such that*

$$Z(\varpi) = \frac{-K^+\alpha}{p - \alpha c^+}. \tag{17.5.4}$$

PROOF. We compute

$$\begin{aligned} Z'(x) &= -1 + \frac{\beta_1 - \beta_2}{\beta_1\beta_2}\,\frac{\beta_2\exp\beta_2 x(\exp\beta_1 x - 1)^2 - \beta_1\exp\beta_1 x(\exp\beta_2 x - 1)^2}{(\exp\beta_1 x - \exp\beta_2 x)^2} \\ &= \frac{[\beta_2(\exp\beta_1 x - 1) - \beta_1(\exp\beta_2 x - 1)]}{\beta_1\beta_2(\exp\beta_1 x - \exp\beta_2 x)^2} \\ &\quad \cdot \frac{[\beta_1\exp\beta_1 x(\exp\beta_2 x - 1) - \beta_2\exp\beta_2 x(\exp\beta_1 x - 1)]}{\beta_1\beta_2(\exp\beta_1 x - \exp\beta_2 x)^2}. \end{aligned}$$

We check easily that for $x > 0$

$$\beta_2(\exp\beta_1 x - 1) - \beta_1(\exp\beta_2 x - 1) < 0$$
$$\beta_1 \exp\beta_1 x(\exp\beta_2 x - 1) - \beta_2 \exp\beta_2 x(\exp\beta_1 x - 1) < 0$$

therefore $Z'(x) \leq 0$. Moreover $Z(0) = 0$ and $Z(+\infty) = -\infty$. This implies the result. □

From (17.5.3) we can assert that

(17.5.5) $$A(a(b), b) - a(b) = \varpi \quad \text{if} \quad A(a(b), b) \leq 0.$$

Going back to the second condition we can write

(17.5.6) $$h - \alpha c^- - (p+h)\frac{\beta_1 \exp\beta_2 b - \beta_2 \exp\beta_1 b}{\beta_1 - \beta_2} = (p - \alpha c^+) \cdot \frac{\exp\beta_1(b-a)(\exp\beta_2\varpi - 1) - \exp\beta_2(b-a)(\exp\beta_1\varpi - 1)}{\exp\beta_1\varpi - \exp\beta_2\varpi}$$

which defines a as a function of b. Let us introduce the number

(17.5.7) $$\varphi(\varpi) = \frac{\beta_1(\exp\beta_2\varpi - 1) - \beta_2(\exp\beta_1\varpi - 1)}{\exp\beta_1\varpi - \exp\beta_2\varpi},$$

then we have

$$\varphi(\varpi) + \beta_1 = \frac{(\beta_1 - \beta_2)(\exp\beta_1\varpi - 1)}{\exp\beta_1\varpi - \exp\beta_2\varpi}; \quad \varphi(\varpi) + \beta_2 = \frac{(\beta_1 - \beta_2)(\exp\beta_2\varpi - 1)}{\exp\beta_1\varpi - \exp\beta_2\varpi},$$

and (17.5.6) becomes after rearranging

(17.5.8)
$$L_\varpi(b-a) = \frac{\alpha(c^+ + c^-)(\beta_2 - \beta_1) + (p+h)[\beta_2(\exp\beta_1 b - 1) - \beta_1(\exp\beta_2 b - 1)]}{p - \alpha c^+},$$

with

(17.5.9) $$L_\varpi(x) = (\exp\beta_1 x - 1)(\varphi(\varpi) + \beta_2) - (\exp\beta_2 x - 1)(\varphi(\varpi) + \beta_1), \ x \geq 0.$$

We can also give another formulation of equation (17.5.8). Recalling the definition of $Z^+(a, b)$ we check easily that (17.5.8) is equivalent to

(17.5.10) $$\frac{Z^+(a,b)}{\exp\beta_1(b-a) - \exp\beta_2(b-a)} = \frac{\varphi(\varpi)(p - \alpha c^+)}{\alpha(\beta_2 - \beta_1)}.$$

Now, we check easily that $\varphi(0) = 0$. From (17.3.13) we see also that $a_0(b)$ is the unique solution of (17.5.10) when $\varpi = 0$. More generally, we state the

Lemma 17.4. *For any $\varpi \geq 0$, there exists a unique $a_\varpi(b)$ solution of* (17.5.10).

PROOF. It is of course equivalent to consider the equation (17.5.8). The function $L_\varpi(x)$ satisfies

$$L'_\varpi(x) = \beta_1(\varphi(\varpi) + \beta_2)\exp\beta_1 x - \beta_2(\varphi(\varpi) + \beta_1)\exp\beta_2 x;$$
$$L"_\varpi(x) = (\beta_1)^2(\varphi(\varpi) + \beta_2)\exp\beta_1 x - (\beta_2)^2(\varphi(\varpi) + \beta_1)\exp\beta_2 x.$$

We note that

(17.5.11) $$\varphi(\varpi) > 0, \ \ \varphi(\varpi) + \beta_2 < 0,$$

hence $L"_\varpi(x) < 0$. Therefore $L'_\varpi(x)$ is decreasing and

$$L'_\varpi(0) = (\beta_1 - \beta_2)\varphi(\varpi), \ \ L'_\varpi(+\infty) = -\infty.$$

It follows that $L'_\varpi(x)$ vanishes at a positive point $\bar{x}$ and

$$L'_\varpi(x) > 0, \forall 0 \le x < \bar{x}, \quad L'_\varpi(x) < 0, \forall\, x > \bar{x}$$

Since $L_\varpi(0) = 0$ and $L_\varpi(+\infty) = -\infty$ the function $L_\varpi(x)$ has a maximum at $\bar{x}$ and has a unique 0 at a point $x^* > \bar{x}$. For $x > x^*$ it is strictly negative, and for $0 < x < x^*$ it is positive. We check easily that the right hand side of (17.5.8) is negative and smaller than $L_\varpi(b)$. Therefore there exists one and only one $a_\varpi(b)$ solution of (17.5.8). □

Similarly, for a given, the pair $b(a)$ and $B(a, b(a))$ is solution of

$$G(a,b) = 0, \quad H^-_{ab}(B) = 0, \tag{17.5.12}$$

and for $B > 0$, we get a simplification, similar to the case $A < 0$ above. From the formula of $G(a,b)$ we can write

$$\begin{aligned} -K^- + \frac{1}{\alpha}(h - \alpha c^-)(b - B) + \frac{1}{\alpha}\frac{h - \alpha c^-}{\beta_1\beta_2}(\beta_1 - \beta_2) \\ \cdot \frac{(\exp\beta_1(B-b) - 1)(\exp\beta_2(B-b) - 1)}{\exp\beta_1(B-b) - \exp\beta_2(B-b)} = 0 \end{aligned} \tag{17.5.13}$$

which leads to $b - B = \pi$ with $\pi > 0$, the solution of

$$Z(\pi) = \frac{-K^-}{h - \alpha c^-}. \tag{17.5.14}$$

Also, from the formula of $H^-_{ab}(B)$ we obtain, using the definition of the function φ, see (17.5.7)

$$L_{-\pi}(a-b) = \frac{\alpha(c^+ + c^-)(\beta_2 - \beta_1) + (p+h)[\beta_2(\exp\beta_1 a - 1) - \beta_1(\exp\beta_2 a - 1)]}{h - \alpha c^-}. \tag{17.5.15}$$

We note that

$$\varphi(-\pi) < 0, \;\; \varphi(-\pi) + \beta_1 > 0. \tag{17.5.16}$$

We see that $L'_{-\pi}(x)$ is positive for $x < \bar{\bar{x}} < 0$ and negative for $x > \bar{\bar{x}}$. We have $L_{-\pi}(0) = 0$ and $L_{-\pi}(-\infty) = -\infty$. Noting that that the right hand side of (17.5.15) is negative and less than $L_{-\pi}(a)$, equation (17.5.15) defines for any $\pi \ge 0$ in a unique way a positive number $b_\pi(a)$. Collecting results, we can state the

Proposition 17.4. *We assume* (17.3.11)*. Then, for ϖ solution of* (17.5.4) *one has*

$$a_\varpi(b) + \varpi < 0 \Longrightarrow a(b) = a_\varpi(b), \;\; A(a(b), b) = a_\varpi(b) + \varpi, \tag{17.5.17}$$

and for π solution of (17.5.14) *one has*

$$b_\pi(a) - \pi > 0 \Longrightarrow b(a) = b_\pi(a), \;\; B(a, b(a)) = b_\pi(a) - \pi. \tag{17.5.18}$$

PROOF. Under the assumptions $a_\varpi(b), a_\varpi(b) + \varpi$ and $b_\pi(a), b_\pi(a) - \pi$ are the unique solutions of (17.5.1) and (17.5.12) respectively. □

17.5.2. PROPERTIES OF $a_\varpi(b)$ AND $b_\pi(a)$. We shall prove the

Proposition 17.5. *We assume (17.3.11). Then $a'_\varpi(0) = b'_\pi(0) = 1$. There exist unique points $b^*_\varpi > 0$ and $a^*_\pi < 0$ such that*

$$a'_\varpi(b^*_\varpi) = b'_\pi(a^*_\pi) = 0.$$

The function $a_\varpi(b)$ increases on $(0, b^*_\varpi)$ and decreases on $(b^*_\varpi, +\infty)$. The function $b_\pi(a)$ decreases on $(-\infty, a^*_\pi)$ and increases on $(a^*_\pi, 0)$. Moreover

$$a_\varpi(b) < a_0(b), \forall \varpi > 0, \quad b_\pi(a) > b_0(a), \forall \pi > 0. \tag{17.5.19}$$

PROOF. We can compute $a'_\varpi(b)$ from (17.5.8). We use also (17.5.8) to combine terms. After rearrangements we obtain
(17.5.20)

$$a'_\varpi(b) = \frac{\beta_1 - \beta_2}{p - \alpha c^+} \cdot \frac{(p - \alpha c^+)\exp\beta_2(b - a_\varpi(b))(\varphi(\varpi) + \beta_1) - (p + h)\beta_1 \exp\beta_2 b + \beta_1(h - \alpha c^-)}{\beta_1 \exp\beta_1(b - a_\varpi(b))(\varphi(\varpi) + \beta_2) - \beta_2 \exp\beta_2(b - a_\varpi(b))(\varphi(\varpi) + \beta_1)}.$$

Note also that

$$1 - a'_\varpi(b) = \frac{p + h}{p - \alpha c^+} \frac{\beta_1\beta_2(\exp\beta_1 b - \exp\beta_2 b)}{\beta_1 \exp\beta_1(b - a_\varpi(b))(\varphi(\varpi) + \beta_2) - \beta_2 \exp\beta_2(b - a_\varpi(b))(\varphi(\varpi) + \beta_1)},$$

from which we get $a'_\varpi(0) = 1$ and $a'_\varpi(b) < 0$ for b sufficiently large.

Let us check that there is a unique point b such that $a'_\varpi(b) = 0$. We call it b to simplify notation. From (17.5.20) we get

$$\exp -\beta_2 a_\varpi(b) = \frac{\beta_1}{\varphi(\varpi) + \beta_1} \frac{p + h - (h - \alpha c^-)\exp -\beta_2 b}{p - \alpha c^+}. \tag{17.5.21}$$

But combining with (17.5.8) which is rewritten as

$$\begin{aligned}(p - \alpha c^+)&\exp\beta_2(b - a_\varpi(b))(\varphi(\varpi) + \beta_1) - (p + h)\beta_1 \exp\beta_2 b \\ &= (p - \alpha c^+)\exp\beta_1(b - a_\varpi(b))(\varphi(\varpi) + \beta_2) - (p + h)\beta_2 \exp\beta_1 b \\ &\quad + (h - \alpha c^-)(\beta_2 - \beta_1),\end{aligned}$$

we have also

$$\exp -\beta_1 a_\varpi(b) = \frac{\beta_2}{\varphi(\varpi) + \beta_2} \frac{p + h - (h - \alpha c^-)\exp -\beta_1 b}{p - \alpha c^+}. \tag{17.5.22}$$

We can then eliminate $a_\varpi(b)$ and obtain an equation for the value of b. We have

$$\begin{aligned}(p + h - (h - \alpha c^-)&\exp -\beta_2 b)(p + h - (h - \alpha c^-)\exp -\beta_1 b)^{-\frac{\beta_2}{\beta_1}} \\ &= \frac{\varphi(\varpi) + \beta_1}{\beta_1}\left(\frac{\varphi(\varpi) + \beta_2}{\beta_2}\right)^{-\frac{\beta_2}{\beta_1}} (p - \alpha c^+)^{1 - \frac{\beta_2}{\beta_1}}\end{aligned} \tag{17.5.23}$$

Let us check that (17.5.23) defines a unique positive value of b. First, the right hand side is a number between 0 and $(p - \alpha c^+)^{1 - \frac{\beta_2}{\beta_1}}$. This follows from the property

$$0 < \left(1 + \frac{x}{\beta_1}\right)\left(1 + \frac{x}{\beta_2}\right)^{-\frac{\beta_2}{\beta_1}} < 1, \forall 0 < x < -\beta_2.$$

On the other hand the left hand side of (17.5.23) decreases in the argument b on $[0, \infty)$ from $(p + \alpha c^-)^{1 - \frac{\beta_2}{\beta_1}}$ to $-\infty$. Since

$$(p + \alpha c^-)^{1 - \frac{\beta_2}{\beta_1}} > (p + \alpha c^-)^{1 - \frac{\beta_2}{\beta_1}},$$

the number b is uniquely defined. We call this number b^*_ϖ. We show that this point is a positive maximum. First we know that $a'_\varpi(0) = 1$. Since b^*_ϖ is the unique point such that $a'_\varpi(b) = 0$, we necessarily have $a'_\varpi(b) > 0, \forall 0 \leq b < b^*_\varpi$. Necessarily $a"(b^*_\varpi) \leq 0$. We cannot have $a"(b^*_\varpi) = 0$. Indeed the point will be an inflection point and we would have $a'_\varpi(b)$ for $b > b^*_\varpi$, close to b. Since, we would have another stationary point different from b^*_ϖ. This is not possible, from the uniqueness of the stationary point. This proves the properties of $a_\varpi(b)$. The property (17.5.19) follows from the fact that $Z^+(a_\varpi(b), b) < 0$, see (17.5.10) and the increasing monotonicity in a of the function. All the properties of $a_\varpi(b)$ have been proven. The properties of $b_\pi(a)$ are proven in a similar way. This concludes the proof. □

We deduce the

Corollary 17.1. *If*

$$a_\varpi(b^*_\varpi) + \varpi \leq 0, \tag{17.5.24}$$

then $a(b) = a_\varpi(b), \forall b > 0$. *Similarly, if*

$$b_\pi(a^*_\pi) - \pi \geq 0, \tag{17.5.25}$$

then $b(a) = b_\pi(a), \forall a < 0$. *If* (17.5.24) *is not satisfied, there exist two and only two values* b^1_ϖ *and* b^2_ϖ *such that*

$$b^1_\varpi < b^*_\varpi < b^2_\varpi, \quad a_\varpi(b^1_\varpi) + \varpi = a_\varpi(b^2_\varpi) + \varpi = 0, \tag{17.5.26}$$

with $b^2_\varpi = +\infty$, *if* $a_\varpi(+\infty) + \varpi \geq 0$. *Then*

$$a(b) = a_\varpi(b), \forall b < b^1_\varpi \text{ and } b > b^2_\varpi. \tag{17.5.27}$$

Similarly, if (17.5.25) *is not satisfied, then there exist two and only two values* a^1_π *and* a^2_π *such that*

$$a^1_\pi < a^*_\pi < a^2_\pi, \quad b_\pi(a^1_\pi) - \pi = b_\pi(a^2_\pi) - \pi = 0 \tag{17.5.28}$$

with $a^1_\pi = -\infty$, *if* $b_\pi(-\infty) - \pi \leq 0$. *Then*

$$b(a) = b_\pi(a), \forall a < a^1_\pi \text{ and } a > a^2_\pi. \tag{17.5.29}$$

PROOF. Clearly $a_\varpi(b) + \varpi < 0, \forall b > 0$ and $b_\pi(a) - \pi > 0, \forall a < 0$. The result follows from Proposition 17.4. We next notice that

$$a_\varpi(0) + \varpi < 0, \quad b_\pi(0) - \pi > 0.$$

The rest of the discussion follows easily. □

The criteria (17.5.24) and (17.5.25) are easy to check, since they involve the value of single numbers.

17.5.3. SEARCH PROCEDURE. We first note that $a_\varpi(+\infty)$ and $b_\pi(-\infty)$ are finite. Indeed

$$\exp -\beta_1 a_\varpi(+\infty) = \frac{p+h}{p-\alpha c^+}\frac{\beta_2}{\varphi(\varpi)+\beta_2}; \tag{17.5.30}$$

$$\exp -\beta_2 b_\pi(-\infty) = \frac{p+h}{h-\alpha c^-}\frac{\beta_1}{\varphi(-\pi)+\beta_1}. \tag{17.5.31}$$

It follows that the curves $a_\varpi(b)$ and $b_\pi(a)$ cross since, setting

$$u(a) = a_\varpi(b_\pi(a)) - a,$$

it varies continuously between $u(0) = a_\varpi(b_\pi(0)) < 0$ and $u(-\infty) = +\infty$ and thus crosses the value 0.

Consider a solution $\hat{a}$,$\hat{b}$ of the system

$$\hat{a} = a(\hat{b}); \quad \hat{b} = b(\hat{a}),$$

and

$$\hat{A} = A(\hat{a}, \hat{b}); \quad \hat{B} = B(\hat{a}, \hat{b}).$$

If $\hat{A} < 0,\ \hat{B} > 0$ then

$$\hat{A} = \hat{a} + \varpi, \quad \hat{B} = \hat{b} - \pi,$$

and also

$$\hat{a} = a_\varpi(\hat{b}), \quad \hat{b} = b_\pi(\hat{a}).$$

If $\hat{A} > 0$, then $\hat{B} > 0$. Therefore

$$\hat{B} = \hat{b} - \pi, \quad \hat{b} = b_\pi(\hat{a}).$$

Since $\hat{a} \neq a_\varpi(\hat{b})$, then $\hat{a} + \varpi > 0$. We must also have $a_\varpi(b^*_\varpi) + \varpi > 0$, so the interval (b^1_ϖ, b^2_ϖ) is well defined (possibly $b^2_\varpi = +\infty$) and

$$b^1_\varpi < \hat{b} < b^2_\varpi.$$

Since $\hat{b} = b_\pi(\hat{a})$ the sign of $b_\pi(a^*_\pi) - \pi$ is not specified. However if $b_\pi(a^*_\pi) - \pi < 0$, then the interval (a^1_π, a^2_π) is well defined (possibly $a^1_\pi = -\infty$) and we have

$$\hat{a} < a^1_\pi \text{or } \hat{a} > a^2_\pi.$$

The first case disappears if $a^1_\pi + \varpi < 0$ (in particular if $a^1_\pi = -\infty$).

Similarly if $\hat{B} < 0$, then $\hat{A} < 0$. So we have

$$\hat{A} = \hat{a} + \varpi, \quad \hat{a} = a_\varpi(\hat{b}).$$

Since $\hat{b} \neq b_\pi(\hat{a})$, then $\hat{b} - \pi < 0$. We must also have $b_\pi(a^*_\pi) - \pi < 0$, so the interval (a^1_π, a^2_π) is well defined (possibly $a^1_\pi = -\infty$) and

$$a^1_\pi < \hat{a} < a^2_\pi.$$

Since $\hat{a} = a_\varpi(\hat{b})$ the sign of $a_\varpi(b^*_\varpi) + \varpi$ is not specified. However if $a_\varpi(b^*_\varpi) + \varpi > 0$, then the interval (b^1_ϖ, b^2_ϖ) is well defined (possibly $b^2_\varpi = +\infty$) and we have

$$\hat{b} < b^1_\varpi \text{ or } \quad \hat{b} > b^2_\varpi.$$

The second case disappears if $b^2_\varpi > \pi$, in particular if $b^2_\varpi = +\infty$.

From the preceding discussion, we can define the following procedure for the search of the pair $\hat{a}, \hat{b}$. We consider a crossing point of the curves $a_\varpi(b)$ and $b_\pi(a)$, denoted by a_0, b_0.

1. If $a_0 + \varpi \leq 0$ and $b_0 - \pi \geq 0$, then $\hat{a} = a_0,\ \hat{b} = b_0$
2. If $a_\varpi(b^*_\varpi) + \varpi \leq 0$ and $b_\pi(a^*_\pi) - \pi \geq 0$, then $\hat{a} = a_0,\ \hat{b} = b_0$
3. If one condition in step 1 and 2 is not satisfied, then we consider two possible scenarios:

–Scenario 1

If $a_\varpi(b^*_\varpi)+\varpi > 0$ and $b_\pi(a^*_\pi)-\pi \geq 0$, then $\hat b = b_\pi(\hat a)$,$b^1_\varpi < b_\pi(\hat a) < b^2_\varpi$,$\hat B = \hat b-\pi$ and $-\varpi \leq \hat a \leq 0,\ 0 < \hat A < b_\pi(\hat a) - \pi$ and

$$K^+ + \frac{1}{\alpha}((h+\alpha c^+)\hat A + \hat a(p-\alpha c^-)) + \frac{1}{\alpha}\frac{p-\alpha c^+}{\beta_1\beta_2}(\beta_1-\beta_2)$$
$$\cdot\frac{(\exp\beta_1(\hat A-\hat a)-1)(\exp\beta_2(\hat A-\hat a)-1)}{\exp\beta_1(\hat A-\hat a)-\exp\beta_2(\hat A-\hat a)} + \frac{1}{\alpha}\frac{p+h}{\beta_1\beta_2}$$
$$\cdot\frac{\beta_1(\exp\beta_1(\hat A-\hat a)-1)(1-\exp\beta_2\hat A)-\beta_2(\exp\beta_2(\hat A-\hat a)-1)(1-\exp\beta_1\hat A)}{\exp\beta_1(\hat A-\hat a)-\exp\beta_2(\hat A-\hat a)} = 0;$$

$$(c^++c^-)(\exp\beta_1(\hat A-\hat a)-\exp\beta_2(\hat A-\hat a)) - \frac{h+\alpha c^+}{\alpha}$$
$$\cdot(\exp\beta_1(\hat A-\hat a)-\exp\beta_2(\hat A-\hat a)-\exp\beta_1(b_\pi(\hat a)-\hat a)+\exp\beta_2(b_\pi(\hat a)-\hat a))$$
$$-\frac{p+h}{\alpha(\beta_1-\beta_2)}\ (\exp(\beta_2\hat A+\beta_1 b_\pi(\hat a))-\exp(\beta_1\hat A+\beta_2 b_\pi(\hat a)))$$
$$\cdot(\beta_1\exp-\beta_1\hat a-\beta_2\exp-\beta_2\hat a)+\frac{p-\alpha c^+}{\alpha}\exp-(\beta_1+\beta_2)\hat a$$
$$\cdot(\exp(\beta_2\hat A+\beta_1 b_\pi(\hat a))-\exp(\beta_1\hat A+\beta_2 b_\pi(\hat a)))=0.$$

–Scenario 2

If $a_\varpi(b^*_\varpi)+\varpi \leq 0$ and $b_\pi(a^*_\pi)-\pi < 0$, then $\hat a = a_\varpi(\hat b)$, $a^1_\pi < a_\varpi(\hat b) < a^2_\pi$, $\hat A = \hat a+\varpi$, $0 < \hat b < \pi$,$a_\varpi(\hat b)+\varpi < \hat B < 0$ and

$$-K^- + \frac{1}{\alpha}((h-\alpha c^-)\hat b + (p+\alpha c^-)\hat B)$$
$$+\frac{1}{\alpha}\frac{h-\alpha c^-}{\beta_1\beta_2}(\beta_1-\beta_2)\frac{(\exp\beta_1(\hat B-\hat b)-1)(\exp\beta_2(\hat B-\hat b)-1)}{\exp\beta_1(\hat B-\hat b)-\exp\beta_2(\hat B-\hat b)}+\frac{1}{\alpha}\frac{p+h}{\beta_1\beta_2}$$
$$\cdot\frac{\beta_1(\exp\beta_1(\hat B-\hat b)-1)(1-\exp\beta_2\hat B)-\beta_2(\exp\beta_2(\hat B-\hat b)-1)(1-\exp\beta_1\hat B)}{\exp\beta_1(\hat B-\hat b)-\exp\beta_2(\hat B-\hat b)} = 0;$$

$$-(c^++c^-)(\exp\beta_1(\hat B-\hat b)-\exp\beta_2(\hat B-\hat b))+\frac{p+\alpha c^-}{\alpha}$$
$$\cdot(\exp\beta_1(\hat B-\hat b)-\exp\beta_2(\hat B-\hat b)-\exp\beta_1(a_\varpi(\hat b)-\hat b)+\exp\beta_2(a_\varpi(\hat b)-\hat b))$$
$$+\frac{p+h}{\alpha(\beta_1-\beta_2)}\ (\exp(\beta_2\hat B+\beta_1 a_\varpi(\hat b))-\exp(\beta_1\hat B+\beta_2 a_\varpi(\hat b)))$$
$$\cdot(\beta_1\exp-\beta_1\hat b-\beta_2\exp-\beta_2\hat b)-\frac{h-\alpha c^-}{\alpha}\exp-(\beta_1+\beta_2)b$$
$$\cdot(\exp(\beta_2\hat B+\beta_1 a_\varpi(\hat b))-\exp(\beta_1\hat B+\beta_2 a_\varpi(\hat b)))=0.$$

If $a_\varpi(b^*_\varpi)+\varpi > 0$ and $b_\pi(a^*_\pi)-\pi < 0$, then one may have either scenario 1 or scenario 2. If scenario 1 takes place then we have the additional restriction on $\hat{a}$: $\hat a < a^1_\pi$ or $\hat a > a^2_\pi$. The first case disappears if $a^1_\pi+\varpi < 0$ (in particular if $a^1_\pi = -\infty$). if scenario 2 takes place, then we have the additional restriction on $\hat b$: $\hat b < b^1_\varpi$ or $\hat b > b^2_\varpi$. The second case disappears if $b^2_\varpi > \pi$, in particular if $b^2_\varpi = +\infty$.

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APPENDIX A

A.1. PROOF OF LEMMAS

A.1.1. PROOF OF LEMMA 4.1.

PROOF. Let us denote by $d(x,y)$ the distance on X. We can assume that $f \not\equiv +\infty$, otherwise $f_n(x) = n$ has the required properties. We then define

$$g_n(x) = \inf_{y \in X} [f(y) + nd(x,y)], \tag{A.1.1}$$

then $g_n(x)$ is uniformly continuous. Indeed

$$|g_n(x) - g_n(x')| \leq nd(x,x').$$

Let next

$$f_n(x) = \min(n, g_n(x)),$$

then $f_n = \min(n, g_n)$ belongs to $C(X)$. Moreover, it is an increasing sequence. We have $f_n(x) \leq f(x)$. Suppose $f(x) < \infty$, then since $g_n(x) \leq f(x)$, we can assert that $f_n(x) = g_n(x)$ for n sufficiently large. Moreover since f is l.s.c. and bounded below, there exists $y_n = y_n(x)$ such that

$$g_n(x) = f(y_n) + nd(x, y_n).$$

Since this expression is bounded above and f is bounded below, one obtains easily that $y_n \to x$, as $n \uparrow \infty$. Since f is l.s.c we have

$$f(x) \leq \liminf\, f(y_n) \leq \liminf\, g_n(x).$$

But also $\limsup g_n(x) \leq f(x)$. Therefore $g_n(x) \uparrow f(x)$. Hence also $f_n(x) \uparrow f(x)$. Suppose now $f(x) = +\infty$. Necessarily $g_n(x) \uparrow +\infty$. Otherwise $g_n(x)$ remains bounded above by a constant A_x, which is impossible. Indeed, considering y_n as above, we have $y_n \to x$, and by lower semi continuity of f, it follows $f(x) \leq A_x$, which contradicts the assumption. Finally if $g_n(x) \uparrow +\infty$ we have also $f_n(x) \uparrow +\infty$, which completes the proof. □

A.1.2. PROOF OF LEMMA 4.2.

PROOF. Without loss of generality, we may assume $l_v(x) = l(x,v) \geq 0$. There exists a sequence $l_v^n(x) = l^n(x,v) \uparrow l(x,v)$ and $l^n(x,v)$ is uniformly continuous and bounded. From assumption (4.3.1) the sequence $\Phi^v l_v^n(x)$ is uniformly continuous and bounded. Now we can write

$$\Phi^v l_v^n(x) = \int P(x,v;d\xi) l^n(\xi,v) = \int\int P(x,v;d\xi) \otimes \delta_v(\varsigma) l^n(\xi,\zeta),$$

therefore we can apply the convergence monotone theorem to claim that

$$\Phi^v l_v^n(x) \uparrow \Phi^v l_v(x),$$

which completes the proof. □

A.1.3. PROOF OF LEMMA 4.3.

PROOF. Firstly, for any fixed x, there exists $\hat{v}(x) \in U$ such that

$$G(x) = F(x, \hat{v}(x)).$$

Indeed, let us consider a sequence v_n such that $F(x, v_n) \to G(x)$. Since v_n is in U, which is compact, we can extract a converging subsequence $v_{n_k} \to v^*$. Since F is l.s.c. we can write

$$F(x, v^*) \leq G(x).$$

Since the reverse is also true, we have equality, So the infimum is attained. Let us now consider a sequence $x_n \to x$ and set $v_n = \hat{v}(x_n)$. We can extract from the sequence x_n, v_na converging subsequence $x_{n_k} \to x, v_{n_k} \to v^*$, Since F is l.s.c. with respect to the two arguments x, v, we can assert that

$$F(x, v^*) \leq \liminf\, F(x_{n_k}, v_{n_k}) = \liminf\, G(x_{n_k}).$$

Therefore

$$G(x) \leq F(x, v^*) \leq \liminf\, G(x_{n_k}).$$

We can always assume that x_{n_k} is itself extracted from a preliminary subsequence, such that $G(x_{n_k}) \to \liminf\, G(x_n)$. Therefore, G is l.s.c. □

A.2. PROOF OF MEASURABLE SELECTION

We prove now Theorem 4.1. Note that the dependence in x of $\hat{v}(x)$ defined in the proof of Lemma 4.3 is not defined. The fact that we can choose a measurable selection is essential in obtaining an optimal feedback to solve optimal stochastic control problems. The result requires multi-valued function theory. We need some concepts and preliminary results. First we have

Lemma A.1. *If f is l.s.c. then $\{x | f(x) \leq c\}$ is closed, for any real c.*

PROOF. Indeed consider a sequence x_n such that $x_n \to x$, and $f(x_n) \leq c$. Since

$$f(x) \leq \liminf\, f(x_n),$$

we have $f(x) \leq c$ and the closedness property is proven. Note that the converse is also true. Indeed, assume $x_n \to x$. Let L be an accumulation point of the sequence $f(x_n)$. There exists a subsequence x_{n_k} such that $f(x_{n_k}) \to L$. Let $\epsilon > 0$ and define

$$L(\epsilon) = \begin{cases} L + \epsilon, & \text{if } L > -\infty \\ -\frac{1}{\epsilon}, & \text{if } L = -\infty \end{cases}$$

Then for k sufficiently large $k \geq k(\epsilon)$ we have $f(x_{n_k}) \leq L(\epsilon)$, therefore from the closedness property $f(x) \leq L(\epsilon)$, hence also $f(x) \leq L$. Since L is an arbitrary accumulation point, necessarily $f(x) \leq \liminf\, f(x_n)$. □

Define $\mathcal{F}(U)$ to be the set of closed subsets of U (including the empty set). These subsets are compact. We equip $\mathcal{F}(U)$ with a metric, the Hausdorff metric as follows (distance between sets). First, if $A \neq \emptyset \in \mathcal{F}(U)$, and $u \in U$, we define the distance of u to A by

$$d(u, A) = \min_{a \in A} d(u, a).$$

Note that

$$|d(u, A) - d(v, A)| \leq d(u, v).$$

If $A = \emptyset$, one defines

$$d(u, \emptyset) = \max_{v,w \in U} d(v, w) = \text{diam } (U).$$

We next define for $A, B \in \mathcal{F}(U)$

$$\begin{aligned}
&\varrho(A, B) = \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, B) \right\}, \text{ if } \ A, B \neq \emptyset \\
&\varrho(A, \emptyset) = \varrho(\emptyset,\text{A}) = \text{diam (U), if } A \neq \emptyset \\
&\varrho(\emptyset, \emptyset) = \varrho(\emptyset, \emptyset) = 0.
\end{aligned}$$

This defines a metric on $\mathcal{F}(U)$ and $\mathcal{F}(U)$ becomes a *compact metric space.*

Consider a sequence A_n of elements of $\mathcal{F}(U)$. One defines

$$\begin{aligned}
\liminf \ A_n &= \{u \in U | \ \limsup \ d(u, A_n) = 0\} \\
\limsup \ A_n &= \{u \in U | \ \liminf \ d(u, A_n) = 0\}
\end{aligned}$$

Obviously $\liminf \ A_n \subset \limsup \ A_n$. Moreover, they are closed sets belonging to $\mathcal{F}(U)$. One can check the useful property

$$\liminf \ A_n = \limsup \ A_n = A \iff \varrho(A_n, A) \to 0.$$

In that case, one uses the definition $A = \lim \ A_n$.

If X is a metric space, a map $\Psi : X \to \mathcal{F}(U)$ is called a *multi valued* map. Since X and $\mathcal{F}(U)$ are metric spaces, we have naturally the concepts of continuous, l.s.c, u.s.c and Borel functions. However, two additional concepts can be defined. These additional concepts have been introduced by K. Kuratowski, see [**30**]. This is why they are called (K) u.s.c or (K) l.s.c. We say that Ψ is (K) u.s.c if

$$x_n \quad \to \quad x \ \Rightarrow \limsup \ \Psi(x_n) \subset \Psi(x).$$

Similarly, one says that Ψ is (K) l.s.c. when

$$x_n \to x \ \Rightarrow \liminf \ \Psi(x_n) \supset \Psi(x).$$

One can prove that if Ψ is both (K) u.s.c and (K) l.s.c. then it is continuous. In fact, the most useful concept is that of (K) upper semi continuity. The reason comes these two essential properties:

(A.2.1) $$\Psi \text{ (K) u.s.c} \quad \Rightarrow \quad \Psi \text{ is Borel}$$

Next

(A.2.2)

If Ψ is (K) u.s.c, and G is an open subset of U, then $\{x \in X | \Psi(x) \subset G\}$ is open.

The second property shows that the concept of (K) u.s.c. reduces to continuity for ordinary functions.

Lemma A.2. *Let U be metric compact and $g : U \to]-\infty, +\infty]$ be l.s.c. Define $g^* : \mathcal{F}(U) \to]-\infty, +\infty]$ by*

(A.2.3) $$g^*(A) = \begin{cases} \min_{a \in A} \ g(a), & \text{if } A \neq \emptyset \\ +\infty, & \text{if } A = \emptyset \end{cases}$$

then g^* is l.s.c.

Proof. The set $\emptyset$ is an isolated point of $\mathcal{F}(U)$. Hence it is sufficient to show that g^* is l.s.c on $\mathcal{F}(U) - \emptyset$. For $A \in \mathcal{F}(U) - \emptyset$, there exists $a \in A$ such that $g(a) = g^*(A)$. Consider now a sequence $A_n \in \mathcal{F}(U) - \emptyset$, and $A_n \to A$. Let $a_n \in A_n$, such that $g(a_n) = g^*(A_n)$. Let us consider a subsequence a_{n_k} such that

$$g(a_{n_k}) \to \liminf\ g(a_n) = \liminf\ g^*(A_n).$$

We can assume that $a_{n_k} \to \bar{a}$ (otherwise we take a converging subsequence). By l.s.c we have

$$g(\bar{a}) \leq \liminf\ g(a_n) = \liminf\ g^*(A_n).$$

Since $d(\bar{a}, a_{n_k}) \geq d(\bar{a}, A_{n_k})$, it follows that $d(\bar{a}, A_{n_k}) \to 0$. Hence $\bar{a} \in \liminf\ A_{n_k}$. But $A_{n_k} \to A$, hence $\liminf\ A_{n_k} = A$. Therefore $\bar{a} \in A$. It follows that

$$g^*(A) \leq g(\bar{a}) \leq \liminf\ g^*(A_n),$$

and thus the property has been proven. □

Lemma A.3. *Let U be metric compact and $g : U \to]-\infty, +\infty]$, l.s.c. One defines for $A \in \mathcal{F}(U)$ and $c \in]-\infty, +\infty]$ the function*

$$G(A, c) = \{u \in A |\ g(u) \leq c\},$$

then G is (K) u.s.c from $\mathcal{F}(U) \times]-\infty, +\infty]$ into $\mathcal{F}(U)$.

Proof. Let $A_n \to A$ and $c_n \to c$, we must prove that

$$\limsup_{n\to\infty} G(A_n, c_n) \subset G(A, c). \tag{A.2.4}$$

Let $y \in \limsup_{n\to\infty} G(A_n, c_n)$. By definition,

$$\liminf_{n\to+\infty} d(y, G(A_n, c_n)) = 0.$$

It follows that there exists $y_{n_k} \in G(A_{n_k}, c_{n_k})$ such that $y_{n_k} \to y$. Note that $y_{n_k} \in A_{n_k}$. Therefore $y \in \liminf\ A_{n_k} = A$. Also $g(y_{n_k}) \leq c_{n_k}$ and since g is l.s.c. we get $g(y) \leq c$. Hence $y \in G(A, c)$. This proves (A.2.4) and the desired result. □

Lemma A.4. *Let U be metric compact and $g : U \to]-\infty, +\infty]$ l.s.c. Let g^* be defined by* (A.2.3). *We define*

$$G^*(A) = G(A, g^*(A)),$$

then G^* is a Borel map from $\mathcal{F}(U)$ to $\mathcal{F}(U)$.

Proof. According to Lemma A.3, G is a Borel map and from Lemma A.2 g^* is also a Borel map. By composition G^* is also a Borel map, which proves the result. □

Lemma A.5. *Let U be metric compact. There exists a map $\sigma : \mathcal{F}(U) - \emptyset \to U$, which is Borel and satisfies*

$$\sigma(A) \in A, \quad \forall A \in \mathcal{F}(U) - \emptyset.$$

Proof. Let us consider a sequence g_n of functions which are uniformly continuous on U and bounded and separates the points of U. This means that if $u \neq v$ then there exists g_n such that $g_n(u) \neq g_n(v)$. There exists such a sequence. Indeed, pick a sequence u_k dense in U. Define

$$F_{kn} = \left\{ v \in U |\ d(u_k, v) \leq \frac{1}{n} \right\}.$$

Consider two sets F_{kn} and $F_{k'n'}$. We define a function $g_{kn;k'n'}$ from U to $[0,1]$, which is continuous (hence uniformly continuous, since U is compact) and satisfies:

$$\begin{aligned} &\text{if } F_{kn} \cap F_{k'n'} \neq \emptyset \text{ then } g_{kn;k'n'} = 1 \\ &\text{if } F_{kn} \cap F_{k'n'} = \emptyset \text{ then } g_{kn;k'n'}(F_{kn}) = 0, g_{kn;k'n'}(F_{k'n'}) = 1 \end{aligned}$$

The set of functions $g_{nk;n'k'}$ is a separating sequence. Indeed if $u \neq v$ there exists F_{kn} and $F_{k'n'}$ such that $u \in F_{kn}, v \in F_{k'n'}$.

To the function g_n we associate g_n^* as in Lemma A.2 and G_n^* as in Lemma A.4. Next define the sequence $H_n : \mathcal{F}(U) - \emptyset \to \mathcal{F}(U) - \emptyset$ by the formula

$$H_n(A) = G_n^*[H_{n-1}(A)]; \quad H_0(A) = A$$

Since $A \neq \emptyset$, one has

$$A = H_0(A) \supset H_1(A) \cdots \supset H_n(A).$$

Since

$$H_n(A) = \left\{ u \in H_{n-1}(A) | \, g_n(u) = \min_{a \in H_{n-1}(A)} g_n(a) \right\},$$

the sets $H_n(A)$ are not empty and compacts. We can assert that

$$\cap_{n=0}^{\infty} H_n(A) \neq \emptyset.$$

Indeed, there exists a sequence $a_n \in H_n(A)$. We can extract a subsequence $a_{n_k} \to a$. Necessarily $a \in \cap_{k=1}^{\infty} H_{n_k}(A) = \cap_{n=0}^{\infty} H_n(A)$.

If $u, u' \in \cap_{n=0}^{\infty} H_n(A)$, then by construction

$$g_n(u) = g_n(u') = g_n^*(H_{n-1}(A)),$$

and since the functions g_n separate the points of U, we have $u = u'$. Therefore $\cap_{n=0}^{\infty} H_n(A)$ contains a single point, which is denoted by $\sigma(A)$. Let us check that

$$H_n(A) \to \{\sigma(A)\}. \tag{A.2.5}$$

Indeed, the sequence $H_n(A)$ being decreasing, we can assert that

$$u \in \cap_{n=0}^{\infty} H_n(A) \Longrightarrow d(u, H_n(A)) = 0, \forall n$$

therefore

$$\cap_{n=0}^{\infty} H_n(A) \subset \liminf_n \, H_n(A) \subset \limsup_n \, H_n(A). \tag{A.2.6}$$

On the other hand, if we consider an element $u \in \limsup_n \, H_n(A)$, there exists a sequence $u_{n_k} \in H_{n_k}(A)$ such that $d(u, u_{n_k}) \to 0$. We have $n_k \leq n_{k+1}$, hence for fixed k, $u_{n_j} \in H_{n_k}$. This implies $u \in H_{n_k}$, therefore $u \in \cap_{n=0}^{\infty} H_n(A)$, which with (A.2.6) implies the result (A.2.5).

Now the functions G_n^* being Borel, the functions H_n are also Borel, $\forall n$. Therefore the function $\nu : \mathcal{F}(U) - \emptyset \to \mathcal{F}(U) - \emptyset$ defined by $\nu(A) = \{\sigma(A)\}$ is Borel. To conclude that σ is Borel, we note that $\sigma = \tau^{-1} \circ \nu$, where $\tau : U \to \mathcal{F}(U) - \emptyset$ is defined simply by $\tau(u) = \{u\}$. The map τ is point to point and continuous. Since U is compact, $\tau(U)$ is also compact, hence closed in $\mathcal{F}(U) - \emptyset$. The inverse is Borel, and therefore σ is Borel, which concludes the proof. □

We now prove Theorem 4.1:

PROOF. Consider the map $F^* : X\times]-\infty,+\infty] \to \mathcal{F}(U)$ defined by

$$F^*(x,c) = \{u \in U \mid F(x,u) \le c\},$$

then F^* is (K) u.s.c. Indeed if $x_n \to x, c_n \to c$, we must show that

$$\limsup F^*(x_n,c_n) \subset F^*(x,c).$$

Let $u \in \limsup F^*(x_n,c_n)$, by definition $\liminf\ d(u,F^*(x_n,c_n)) = 0$. There exists a subsequence $u_{n_k} \in F^*(x_{n_k},c_{n_k})$, such that $u_{n_k} \to u$. This implies

$$F(x_{n_k},c_{n_k}) \le c_{n_k}.$$

Since F is l.s.c., we obtain $F(x,u) \le c$, hence $u \in F^*(x,c)$. Define next

$$F^*(x) = F^*\left(x, \inf_{u\in U} F(x,u)\right) : X \to \mathcal{F}(U).$$

Since $F^*(x,c)$ is Borel and $x \to \inf_{u\in U} F(x,u)$ is l.s.c., the map $F^*(x)$ is Borel. Note that

$$F^*(x) = \left\{u \in U\ |F(x,u) = \inf_{v\in U} F(x,v)\right\}.$$

Moreover $F^*(x) \ne \emptyset,\forall$x, hence $F^* : X \to \mathcal{F}(U) - \emptyset$. The map $\hat{v}(x) = \sigma(F^*(x))$ is Borel from X to U and satisfies $\hat{v}(x) \in F^*(x)$. Therefore we have

$$F(x,\hat{v}(x)) = \inf_{v\in U} F(x,v), \forall x,$$

which completes the proof. □

A.3. EXTENSION TO U NON COMPACT

We now prove Theorem 4.2

PROOF. For each fixed x, we can restrict the set of controls to $U \cap \{|v| \le \gamma(x)\}$, which is compact, hence the minimum is attained. To prove that $G(x)$ is l.s.c. take a sequence $x_n \to x$ and let v_n be the corresponding minimum. We have

$$|v_n| \le \gamma(x_n),$$

which remains bounded, by the assumption. So the sequence v_n remains in a compact set. We can extract a subsequence x_{n_k}, v_{n_k} which converges towards x, v^*

Since F is l.s.c. in both arguments we have

$$F(x,v^*) \le \liminf F(x_{n_k},v_{n_k}) = \liminf G(x_{n_k}).$$

It is possible beforehand to assume that

$$G(x_{n_k}) \to \liminf G(x_n),$$

hence

$$G(x) \le \liminf G(x_n),$$

which proves that $G(x)$ is l.s.c. Next, consider the subset $X_N = \{x \in X | d(x_0,x) \le N\}$ where d is the distance in X. For $x \in X_N$ we can restrict the set of controls to

$$U_N = U \cap \{|v| \le \sup_{x\in X_N} \gamma(x)\},$$

which is compact.

Therefore, according to Theorem 4.1 there exists a Borel map $\hat{v}_N(x) : X_N \to U_N$ such that

$$G(x) = F(x,\hat{v}_N(x)), \forall x \in X_N.$$

We next define

$$\hat{v}(x) = \begin{cases} \hat{v}_1(x), & \text{if } x \in X_1 \\ \cdots & \\ \hat{v}_N(x), & \text{if } x \in X_N - X_{N-1} \end{cases}$$

The map $\hat{v}(x)$ is Borel and satisfies $G(x) = F(x, \hat{v}(x)), \forall x$. The proof has been completed. □

A.4. COMPACTNESS PROPERTIES

Let us consider a sequence of functions $u_\alpha(x)$, where $x \in R^{n\cdot}$. We assume (for $\alpha \to 1$)

$$|u_\alpha(x)| \leq C_M, \ \ |Du_\alpha(x)| \leq C_M, \ \forall x \text{ such that } |x| \leq M,$$

then there exists a subsequence, still denoted $u_\alpha(x)$ which converges uniformly on the ball of radius M, to a function $u(x)$ such that

$$|u(x)| \leq C_M, \ \ |Du(x)| \leq C_M, \ \forall x \text{ such that } |x| \leq M$$

and

$$\sup_{|x| \leq M} |u_\alpha(x) - u(x)| \to 0, \text{ as } \alpha \to 1.$$

The convergence of α to 1 can be replaced by a convergence to any fixed number. When $n = 1$, this is the Ascoli-Arzelà theorem. In general this result expresses a compactness property of a set of functions.